

Computer-Based Training Tool for Signals and Systems Exercises*

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A computer-based training tool has been developed for the Signals and Systems undergraduate course for students. It consists of an automatic problem generator and an automatic self-assessment system that can be used by the students as many times as they want. The students use the tool to generate a numerical variation of a problem, solve it, and use the tool again to answer a series of questions that are automatically assessed. The proposed problem comprises the continuous-time-domain characterization of a linear and time invariant system and its Fourier and Laplace domain analysis. The tool was used during the academic year 2009/2010 by a group of 19 students, and their grades in a final exam were compared with those of a similar group of 17 students who did not use the tool. On a 0–10 scale, the mean and standard deviation of the grades of students who used the tool were 6.5 and 1.8, respectively, whereas the mean and standard deviation of the grades of the student who did not use this tool were 5.1 and 2.6, respectively. The analysis of variance of both sets of marks yields to a $p = 0.05$ value. The tool can also be used by the teacher as a computer-based assessment system.

Keywords: computer-based assessment; automatic problem generation; self-learning

1. Introduction

Computer-based assessment (CBA) and web-based assessment have become common tools at all levels of educational [1]. Many learning management systems (LMS) have been developed in recent years to generate quizzes intended for instructional or assessment purposes easily [2–5]. The core of most of these tools is a large and high-quality item bank for any topic to be assessed. Several systems have been developed in order to deliver individualized quizzes, exercises, or tasks to students depending on their performance and/or interest [1, 6–9]. A challenge faced by these tools is to generate the item bank. Over the last decade, great effort has been devoted to develop automatic item generators and exercises generators [8–11]. However, many of these generators use a database where the variations in a given question are stored, so it is a hard task to create variations of the same exercise, solve them, and store the input data and its corresponding solutions [1, 9–13]. Numerical item variations could be the easiest to produce because random numbers can be generated and inserted into the question and the corresponding answer could be easily calculated from the random numbers [1, 8, 9, 14, 15]. This can be done when the item answer is directly related through a simple equation to the random generated data. Some generators that use this strategy allow the definition of ranges for the generated random numbers, but they usually do not allow the addition of relations between different parameters, they do not allow the construction of several questions

based on the same wording of the exercise and the same set of parameters, and they usually do not allow a piece of programming to obtain the solution (they usually just allow an equation relating the random numbers for producing the right solution) [2–4].

There are some systems that allow the formulation and assessment for more than one question for each automatically generated item or exercise [14, 15]. However, the item parameters are randomly generated, so a student cannot repeat the same numerical variation of an exercise. It must be noticed that there are many parameterized exercises that are qualitatively different (and their resolutions could imply quite different difficulty levels) depending on the actual values of the parameters, as is the case for the problem presented in the following section.

Here, a method to generate automatically a large set of numerical variations of the same exercise is proposed [16]. A quite general class of problem in the Signals and Systems undergraduate course is selected: the analysis of a continuous-time linear and time invariant (LTI) system, described by a linear differential equation with constant coefficients (LDECCs). An application has been developed to generate the numerical values of the LDECCs that describe the LTI system from any input number.

The method is based on the using modulo operation (which gives the remainder of the division of two numbers) applied to any input number. This modulo operation is applied with different divisors

in order to get some integer values that will be used to generate the constant coefficients of the differential equation. When the input number is any identification number of a student (i.e., his/her passport number, national identity card number, etc.), the application becomes a personalized work assignment and assessment tool: each student has to solve a different exercise. The idea can be easily applied to any kind of parameterized exercises. The analysis of LTI systems described by an LDECC example is chosen because LDECCs model most of the systems that can be found in engineering and scientific applications.

The main objective of this work is to evaluate the effectiveness of the presented application as a learning tool. A group of students were asked to voluntarily use the application. The results of these students in a final exam were compared with the results of a second group of students who were not informed about the application.

Following this introduction, the context, exercise, and assessment questions are presented. Afterwards, the method for evaluating the influence of the application is described, the results are presented, a discussion of the results is given, and some concluding remarks are made.

2. Presentation

Signals and Systems appears as a main topic in many engineering disciplines. An important question in this course is to analyze and classify (if possible) the frequency response behavior of an LTI system. Here, the attention is focused on the first- and second-order continuous-time LTI systems described by an LDECC. In fact, after revising the final exams held by EUIT de Telecomunicación at the Universidad Politécnica de Madrid during the last five years, the author noticed that most of the exams contain a question where the students are asked to derive the transfer function (and to draw the pole-zero diagram), the impulse response, and/or the frequency response function (including its classification), and a block diagram representation for a system described by an LDECC. This is an analysis exercise. The system is defined and the student should analyze the behavior of the system in the time and frequency domain, and he/she should relate this behavior to the different representations of the system: the LDECC given as a description of the system, the system function, the pole-zero diagram, or a block diagram of the system.

2.1 Exercise

A statement, like the one shown below includes most of the questions related to continuous-time

systems analysis that is studied in an introductory Signals and Systems course:

Exercise 1

The input $x(t)$ and the output $y(t)$ of a causal LTI system are related by the differential equation:

$$\begin{aligned} a_2 \frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) \\ = b_2 \frac{d^2 x(t)}{dt^2} + b_1 \frac{dx(t)}{dt} + b_0 x(t). \end{aligned} \quad (1)$$

- Find the system function and draw its pole-zero diagram.
- Depict a block diagram representation of the system.
- Based on the pole-zero diagram, sketch the impulse response indicating its qualitative behavior.
- Find the impulse response of the system.
- Find the frequency response of the system.
- Sketch the magnitude of the frequency response and classify it (if possible) as an approximation to an ideal filter. Indicate some characteristic parameters as cut-off frequency(s), maximum value of the magnitude, frequency for which that maximum is reached . . .

Wording variations of this exercise can be found in many Signals and Systems textbooks (see for example [17–19]). Note that usually only one or two of the questions are asked at once (depending on the chapter of the book we are consulting). Then, the proposed class of exercise can be understood as one long exercise that covers many aspects of the methods used to analyze the LTI systems behavior (in the time and frequency domains). By solving this kind of exercise, students should learn to relate the behavior of the impulse response and the frequency response amplitude to the pole-zero diagram, to the block diagram, or to the system function. The exercise solution will be quantitatively and qualitatively different depending on the particular set of coefficients. Each selection of $a = [a_2, a_1, a_0]$ and $b = [b_2, b_1, b_0]$ yields a numerically different problem. However, an arbitrary or random choice of those coefficients frequently yields meaningless systems (non-stable systems, for example). Then, the algorithms used to produce numerical variations, typically used in many automatic item generators, are not valid [1, 2, 8, 14, 15].

In our case and for the given problem, the coefficients will be selected in such a way that they satisfy the following criteria:

- The system should be stable (in order to have a frequency response) and causal (in order for the system to have a physical meaning, as a mechan-

ical damper or a passive electric circuit). This is satisfied if the coefficients a_i are real positive numbers (a_2 could be zero) [17].

- C2 The system should represent an approximation to one of the following ideal filters: low-pass, high-pass, band-pass, band-stop, or all-pass filter.
- C3 All the coefficients should be not too high integers (up to 3 digits). Then the required calculations will be quite easy. The purpose is that students will learn the mechanical steps they must follow to obtain the particular solution.
- C4 A large enough set of different problems should be generated. The purpose of this condition is that each student that attends a particular course will have enough numerically different exercises to be solved.
- C5 A single seed is enough as an input to the algorithm in order to generate a particular set of coefficients (a , b).

The Appendix describes details of how to obtain the set of coefficients (a , b) [16].

The teacher can give the same set of seeds to all the students, so they solve the same set of exercises, or the teacher could assign a different set of seeds to each student, so each student solves a different set of numerically different exercises. In addition, students can use the algorithm and the self-assessment test to practice as many times as they want. They should solve the exercise (with different input numbers) until they are sure that they can solve any other numerically different example.

2.2 Self-assessment questions

The developed application automatically assesses a set of proposed questions related to Exercise 1. For a given seed, the application calculates the corresponding a and b vectors. The answers to the following questions are easily programmable from the seed, as shown in Table 1 and Equations (2)–(7) (see Appendix).

The proposed questions are listed below. They refer to the tasks (a) to (f) in Exercise 1.

- (a.1) What is the order of the numerator of the system transfer function?
- (a.2) What is the order of the denominator of the system transfer function?
- (a.3) In which region are the poles of the system transfer function located?
- (a.4) Calculate the poles and zeros of the system transfer function.
- (b.1) Which is the minimum number of integrators necessary to obtain a block diagram of the system?
- (c.1) What is the behavior (for $t > 0$) of the system impulse response?

- (d.1) What is the value of the impulse response for $t \rightarrow 8$?
- (d.2) What is the value of the impulse response for $t \rightarrow 0^+$?
- (d.3) Which of the following functions corresponds to the impulse response?
- (i) $c_1 \exp(-c_2 t) u(t)$
- (ii) $c_0 \delta(t) - c_1 \exp(-c_2 t) u(t)$
- (iii) $c_1 \exp(-c_2 t) \sin(-c_3 t) u(t)$
- (iv) $c_0 \delta(t) - c_1 \exp(-c_2 t) \cos(-c_3 t) u(t)$
- (v) $c_1 \exp(-c_2 t) \sin(-c_3 t) u(t) + c'_1 \exp(-c'_2 t) \cos(-c'_3 t) u(t)$
- (vi) $c_0 \delta(t) + c_1 \exp(-c_2 t) \sin(-c_3 t) u(t) + c'_1 \exp(-c'_2 t) \cos(-c'_3 t) u(t)$
- (vii) $c_1 \exp(-c_2 t) u(t) + c_3 \exp(-c_4 t) u(t)$
- (viii) $c_0 \delta(t) + c_1 \exp(-c_2 t) u(t) + c_3 \exp(-c_4 t) u(t)$
- (ix) Other expression.

When answering this question, it is assumed that the coefficients of the expressions (i)–(viii) have non-zero values. Once a selection is made, the values of the coefficients are also asked for.

- (e.1) What amplification/reduction gives the system a constant entry?
- (e.2) What is the maximum value of the frequency response amplitude?
- (e.3) For which frequency does the frequency response amplitude reach its maximum value?
- (f.1) What ideal filter does the frequency response of the system approximate?
- (f.2) What is the 3-dB cut-off angular frequency (low or high-pass cases) or frequencies (band-pass or band-stop filter) of the system frequency response?
- (f.3) From the straight-line approximations to the Bode diagram, what is the slope (in dB/decade) for frequencies well below the (lower) cut-off frequency?
- (f.4) From the straight-line approximations to the Bode diagram, what is the slope (in dB/decade) for frequencies well above the (higher) cut-off frequency?

For most of these questions a numerical answer is expected. For qualitative questions (for example, a.3, c.1, d.3 . . .), a single choice selection among several items is expected. Figure 1 shows an example of an intermediate screenshot of the application. Once students have answered the questions on a screen, they can click the 'Assess' button. Then, the 'Assess' button is changed to a 'Next' button and the correct answers together with the student answers and 'Ok' or 'Wrong' messages are shown (see Fig. 2). Some hints are also included for wrong answers. This process is repeated until the end of the problem where a global grade for the attempt is shown together with 'Restart' and 'Exit' buttons.

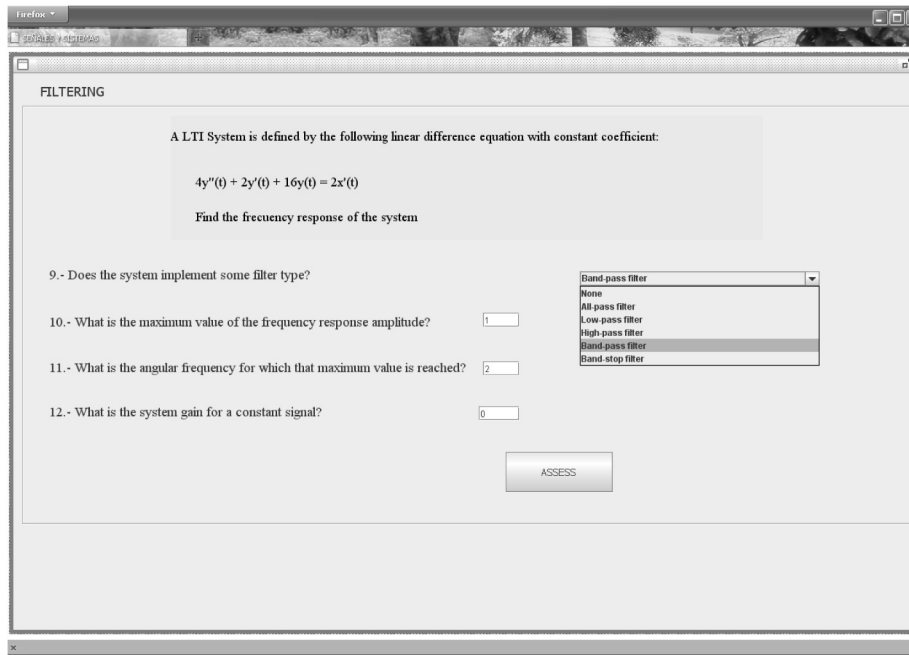


Fig. 1. Screenshot of application where student can select or introduce answers.

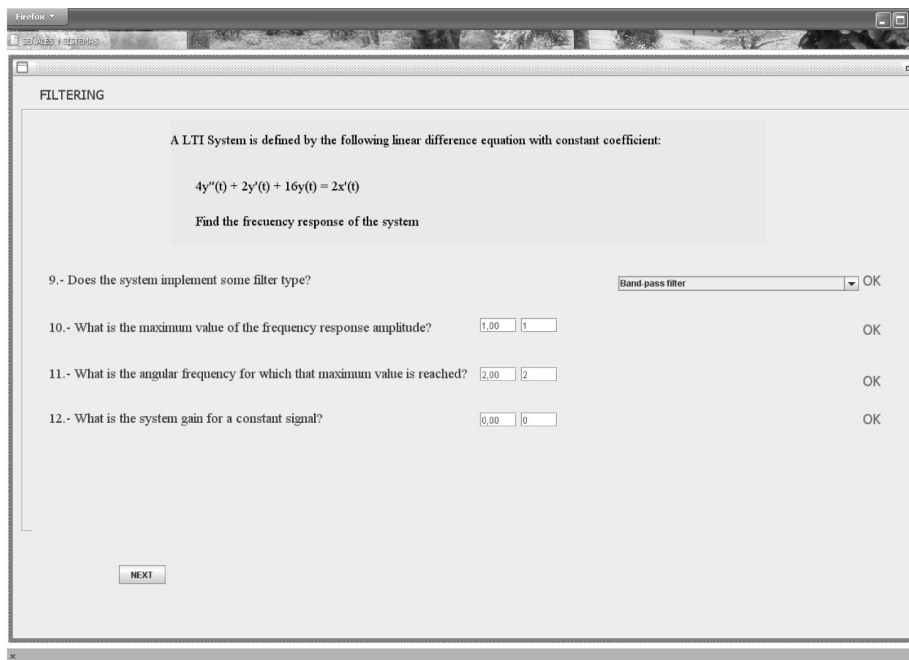


Fig. 2. Screenshot of application after clicking on 'Assess' button.

3. Methodology

This study compares the results obtained by two groups of students arbitrarily allocated. The first group was invited to use the application, and the second group was not informed about the application. Signals and Systems at EUIT de Telecomunicación at the Universidad Politécnica de Madrid is a basic and common course in the second academic

year for all the students. The contents of this course are as follows:

- (i) Introduction to signals
- (ii) Introduction to systems: definition, properties, and time domain LTI systems analysis
- (iii) Continuous-time signal and systems Fourier analysis. Laplace transform analysis
- (iv) Discrete time signal and systems Fourier analysis. Z transform analysis.

Over the last seven years, a mean of 480 ± 40 students enrolled in this matter. These students were divided into eight groups, and each group was assigned to a different teacher. All the groups followed the same teaching methodology and the same evaluation method. The common evaluation method requires a minimum score of 5 (on a 0–10 scale) to pass the subject and it consists of two parts:

1. 50% of the final score is obtained through a continuous evaluation method: homework, classroom exercises, and tests taken via Moodle [2] and individual and group works. Moodle tests are generated from the same item bank for all the students. Exercise and works are proposed by all the teachers.
2. 50% of the final score is obtained through a common final exam. The final exam is divided into three parts: a theoretical true/false test (20% of the final score) and two practical exercises (15% of the final score for each one), where one of them is dedicated to the analysis of a continuous-time LTI system (typically a wording and numerical variation of Exercise 1 where a, b, and f questions always appear).

The final exam is proposed and assessed by the panel of course teachers. During the academic course 2009/2010, the author participated only in lectures and practical sessions and not in the proposal or the assessment of the exercise dedicated to the analysis of continuous-time LTI systems. Both the proposal and the assessment of this exercise were carried out by teachers who had no knowledge of the present experiment (in particular, they did not know that a group of student had used the application presented here).

In the group assigned to the author, there were 47 students who regularly assisted in the theoretical sessions. These students were arbitrarily divided in two subsets: 24 of them were asked to use the application (A group in the following) and solve 5 to 10 times the exercises presented by the application when the first ten natural numbers were used as a seed. The remaining 23 students (B group) were not informed about the application. Four students in the A group did not use the application in the end, nor did they make the final exam, and one student from this A group used the application but did not make the final exam. On the other hand, 17 students in the B group made the final exam. In this study, only the results of students that took the final exam were considered.

The mean value (M) and standard deviation (S) of the scores in the 0–10 range obtained by the students of the two groups were compared. An analysis of variance (ANOVA) was made to decide

if the mean values were statistically different or not. The Cohen effect size was also analyzed [20].

4. Results

After the final exam, exercises were assessed in a masked way, the author extracted the scores obtained in the exercise devoted to the analysis of a continuous-time LTI system for both groups of students. The mean value and standard deviation of the scores obtained by the students of the two groups were $M_A = 6.5$, $S_A = 1.8$, $M_B = 5.1$, and $S_B = 2.6$, where the subscripts refer to the A and B groups, respectively. These values yielded an effect size of 0.67 that was a medium effect size [20]. The analysis of the variance of the two sets of data was performed, assuming they have the same mean. An F value of 4.0 was obtained with a probability $p = 0.05$ of having an F value greater than 4.0. This shows that the mean score obtained by students of A group was statistically higher than the mean score obtained by students of B group. On the other hand, the scores of the B group students showed higher standard deviation, showing a great variation in performance of these students in the final exam.

Histograms of the scores for both the groups are compared in Fig. 3. It can be seen that the columns corresponding to the histogram of the A group are higher than those of the B group for ‘acceptable,’ ‘very good,’ and ‘excellent’ scores. However, for ‘poor’ and ‘marginal’ scores, the columns corresponding to histograms of the B group are equal or higher than those of the A group. So, qualitatively, the scores of the A group were better than those of the B group. The histogram of the B group also explains the large standard deviation S_B : most students of the B group obtained a ‘very good’ or ‘poor’ score, while few of them obtained ‘acceptable’ or ‘marginal’ scores.

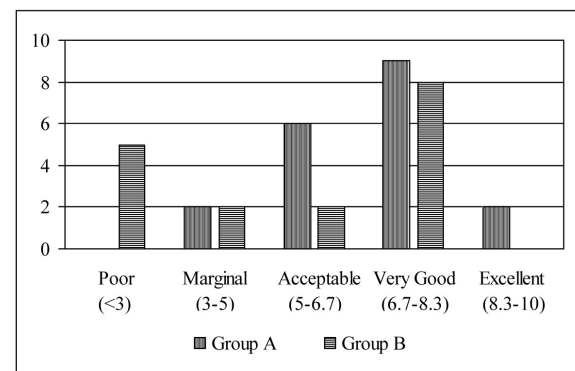


Fig. 3. Histograms of students’ scores in the final exam exercise corresponding to continuous-time LTI system analysis. Group A students used the application, while group B students did not use it.

5. Discussions

It is quite difficult to evaluate the effect of intervention in an educational environment [15]. However, there are thousands of reported experiments and studies about the use of computer- and web-based tools [21]. Online conditions and blended learning shows advantages over purely face-to-face settings. Here, a new experience of introducing a computer-based tool that automatically generates and assesses thousands of variations of a typical exercise in Signals and Systems is presented. Although the number of students involved in this experience is reduced, the results obtained in this case study support the idea that computer-based learning is an effective learning tool when adequately used. Specifically, in the context of the undergraduate Signals and Systems course, the students who used the developed application obtained a mean score 1.4 points higher (in a 0 to 10 scale) than those students who did not use it. This score corresponds to the final exam exercise that evaluates the students' ability to analyze continuous-time LTI systems.

The arbitrary selection of the reduced group of students who had the potential advantage of accessing the application presented here could lead to an ethical rejection of the experiment, but the author considers that because of the evaluation method used, the influence of this potential advantage is small in the overall evaluation process: the students who had access to the application obtained a mean score that was about 15% higher in a part of the evaluation that represented 15% of the total marks. That is, this advantage could represent the final scores of this group of students to be around 2% higher than those of the control group. From the author's knowledge, during the last ten years, none of the students had failed the Signals and Systems course at EUIT de Telecomunicación with a score 2% lower than the minimum required. So the author does not find ethical reasons to reject the experiment.

The algorithm described in the Appendix automatically generates up to 2200 numerical variations of Exercise 1, satisfying quite restrictive criteria (C.1)–(C.5) from a single seed. An application that includes this algorithm and that automatically assesses the set of questions described in Section 2 was implemented. The self-assessment questions varied from very simple ones (they could be answered by simple inspection of the LDECC) to more difficult questions where students had to do some manipulations and mathematical calculations to find the correct answer. Algorithms based on the same idea (the use of modulo operation) are being developed to include time domain analysis of LTI systems (i.e., exercises for training continuous-

and discrete-time convolution) and frequency domain analysis of discrete-time LTI systems [22].

When intended for personalized assignments or for assessment purposes, the algorithm allows the easy selection of sets of seeds in such a way that all the students are required to analyze the same type of systems. For example, the seeds $N_0 = 1 + 10 \times L$ with integer L yield to different first-order low-pass filters, or $N_0 = 3 + 5 \times L$ yields to different second-order band-pass filters. In this way, all the students are required to solve the exercise with the same mathematical workload and with the same conceptual difficulty. In addition, it is harder to cheat than when all students solve exactly the same numerical variation, and the instructor workload is reduced to check the application grading.

6. Conclusions

An application for training in the analysis of continuous-time LTI systems is presented. An algorithm has been developed to generate thousands of numerical variations of a complex exercise automatically. These numerical variations satisfy very restrictive conditions (stated in Section 2.1). A set of questions to easily assess and grade the works is also proposed. The application can be used by students in the training mode to solve as many numerically different exercises as they want: they just have to introduce a new input seed to obtain a new exercise.

The usefulness of the application as a learning tool was evaluated during the 2009/2010 academic year with a reduced number of students. Students who voluntarily used the application obtained higher mean scores in a final exam exercise than those students who were not informed about this application. The difference in the results obtained by these groups of students was statistically significant with a $p = 0.05$ value.

The instructor could also use the application to automatically assess and grade the students' ability to analyze continuous-time LTI systems.

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References

1. M. Thelwall, Computer-based assessment: a versatile educational tool, *Computers & Education*, 34, 2000, pp. 37–49.
2. Moodle.org, *Open-Source Community-Based Tools for Learning*, <http://www.moodle.org>, [Accessed July 20, 2010].
3. Questionmark, <http://www.questionmark.com/uk/index.aspx>, [Accessed July 20, 2010].
4. Openmark, <https://openmark.dev.java.net/>, [Accessed July 20, 2010].
5. A. García-Beltrán and R. Martínez, Web assisted self-assessment in computer programming learning using AulaWeb, *International Journal of Engineering Education*, 22(5), 2006, pp. 1063–1069.

6. E. Guzmán and R. Conejo, Self-assessment in a feasible, adaptive web-based testing system, *IEEE Transactions on Education*, **48**(4), 2005, pp. 688–695.
7. L. Malmi, V. Karavirta, A. Korhonen, J. Nikander, O. Seppala and P. Silvasti, Visual algorithm simulation exercise system with automatic assessment: TRAKLA2, *Informatics in Education*, **3**(2), 2004, pp. 267–288.
8. M. Cupic and Z. Mihajlovic, Computer-based knowledge, self-assessment and training, *Internal Journal of Engineering Education*, **26**(1), 2010, 111–125.
9. P. Brusilovsky and S. Sosnovsky, Individualized exercises for self-assessment of programming knowledge: an evaluation of quizpack, *Journal of Educational Resources in Computing*, **5**(3), 2005, p. 6.
10. M. Arendasy, M. Sommer, G. Gittler and A. Hergovich, Automatic generation of quantitative reasoning items: a pilot study, *Journal of Individual Differences*, **27**(1), 2006, pp. 2–14.
11. M. J. Gierl, J. Zhou and C. Alves, Developing a taxonomy of item model types to promote assessment engineering, *The Journal of Technology, Learning, and Assessment*, **7**(2), 2008, pp. 1–51.
12. L. Weyten, P. Rombouts and J. De Maeyer, Web-based trainer for electrical circuit analysis *IEEE Transactions on Education*, **52**(1) 2009, pp. 185–189.
13. C. R. Smaill, The implementation and evaluation of OASIS: A web-based learning and assessment tool for large classes, *IEEE Transactions on Education*, **48**(4), 2005, pp. 658–663.
14. J. A. González and P. Muñoz, e-status: an automatic web-based problem generator—applications to statistics, *Computer Applications in Engineering Education*, **14**(2) 2006, pp. 151–159.
15. J. A. González, L. Jover, E. Cobo and P. Muñoz, A web-based learning tool improves student performance in statistics: a randomized masked trial, *Computers & Education*, 2010. doi: 10.1016/j.compedu.2010.03.003.
16. J. C. G. de Sande, Automatic generation of signals and systems exercises, *Proceedings of ICERI2010 Conference*, 15–17 November 2010, Madrid, Spain.
17. A. V. Oppenheim, A. S. Willsky and S. H. Nawab, *Signals and Systems*, Prentice-Hall, New Jersey, 1997.
18. J. G. Proakis and D. G. Manolakis, *Digital Signal Processing*, Prentice-Hall, New Jersey 2007.
19. M. J. Roberts, *Signals and Systems: Analysis using Transform Method and Matlab*, McGraw-Hill, 2004.
20. J. Cohen, *Statistical Power Analysis for the Behavioral Sciences*, 2nd edn, Lawrence Erlbaum Associates, Hillsdale, NJ, 1988.
21. U.S. Department of Education, Office of Planning, Evaluation, and Policy Development, *Evaluation of Evidence-Based Practices in Online Learning: A Meta-Analysis and Review of Online Learning Studies*, U.S. Department of Education, Washington, DC, 2009.
22. J. Montes, J. Deza and J. C. G. de Sande, Signals and systems students' trainer. Accepted for presentation at *EDU-LEARN2011 Conference*, July 2011, Barcelona, Spain.

Appendix

Algorithm for generating coefficients

The starting point of the algorithm is an input integer number, seed N_0 , as follows.

The second step is to calculate the N_0 modulo n values for $n = 2$ ($m2$), $n = 3$ ($m3$), $n = 5$ ($m5$), $n = 7$ ($m7$), and $n = 11$ ($m11$).

The third step is to decide which kind of filter (low-pass, high-pass . . .) will be generated. $m5$, that can take 0, 1, 2, 3, or 4 value, is used for this purpose. In the case of all-pass (AP), low-pass (LP), and high-pass (HP) filters, $m2$ is used to select the first-or second-order filter. For the case of band-pass (BP) and band-stop (BS) filters, that will be second-order filters for Exercise 1. Different values of $m2$ modify the amplitude of the frequency response. Table 1 shows the proposed correspondence between the filter type and the order and the values of these two numbers.

The fourth step is to build the first- and second-order Butterworth low-pass approximations to generate a low-pass prototype: $a_{p1} = [0, 1, 1]$ (first-order case) or $a_{p2} = [1, \sqrt{2}, 1]$ (second-order case) and $b_p = [0, 0, 1]$. The 3-dB cut-off frequency for these prototypes is 1 rad/s.

Finally, a frequency transformation is applied to the prototype in order to obtain the a and b vectors that correspond to the proposed filter [18].

Following these steps, conditions C.1 and C.2 are automatically fulfilled. The trick of the proposed algorithm is to appropriately select the new cut-off frequency (or central frequency and bandwidth) in order to satisfy the C.3 and C.4 conditions.

Table A1. Type and order of generated filters for each combination of $m2$ and $m5$ values. This is an arbitrary selection proposed by the author

$m5$	Filter type	$m2$	Filter order
0	All-pass	0	First order
		1	Second order
1	Low-pass	0	First order
		1	Second order
2	High-pass	0	First order
		1	Second order
3	Band-pass	0/1	Second order
4	Band-stop	0/1	Second order

For the cases of low-pass and high-pass filters, the proposed selection of the cut-off angular frequency is

$$\omega_c = \frac{(1 + m11)}{2} (\sqrt{2})^{m2}. \quad (2)$$

In such a way, we obtain 11 different semi-integer or integer values of the cut-off frequency for the case of the first-order filter ($m2 = 0$) that span from $1/2$ to $11/2$. For the case of the second-order filters ($m2 = 1$), we obtain the same 11 values multiplied by $\sqrt{2}$ for the angular cut-off frequency. After the application of the frequency transformation, the coefficients that define the filter (a_t, b_t) are integers or semi-integers. Finally, the proposed coefficients are calculated as follows:

$$\begin{aligned} a &= 2^k (2 + 3m3) a_t \\ b &= 2^k (1 + m7) b_t. \end{aligned} \quad (3)$$

The factor 2^k with $k = 0$ (when all the elements of a_t and b_t are integers) or $k = 1$ (if there is some semi-integer element in a_t or b_t) assures that all the coefficients will be integers and the factors $(2 + 3m3)$ and $(1 + m7)$ modify the amplitude of the frequency response of the system. The largest integer that could appear as an element of a or b is $7 \times 2 \times (\frac{11}{2} \sqrt{2})^2 = 847$. It must be noted that the factor $(1 + m7)/(2 + 3m3) = 1$ when $m7 = 1$ and $m3 = 0$, and when $m7 = 4$ and $m3 = 1$ so $2 \times 3 \times 7 \times 11 \times 2 - 2 \times 2 \times 11 = 880$ numerically different exercises are generated.

For the cases of band-pass and band-stop filters, the proposed selection of the central angular frequency and bandwidth is as follows:

$$\begin{aligned} \omega_0 &= \frac{1 + m11}{2} \\ B &= \omega_0 \left(\frac{m3 + 1}{4} \right). \end{aligned} \quad (4)$$

After the transformation of the prototype filter into a band-pass or band-stop filter, the obtained (a_t, b_t) vectors are modified as follows:

$$\begin{aligned} a &= 2^k (1 + 3m2) a_t \\ b &= 2^k (1 + m7) b_t. \end{aligned} \quad (5)$$

Again, the factor 2^k with $k = 0, 1, 2$, or 3 , depending on the values of a_t and b_t , is introduced to assure that all the elements in a and b are integers. The largest possible element of the generated a and b vectors is $8 \times 4 \times (11/2)^2 = 968$. It must be noted that the factor $(1 + m7)/(1 + 3m2) = 1$ when $m7 = m2 = 0$, and when $m7 = 3$ and $m2 = 1$. In this way the number of numerically different filters obtained following the described method is $2 \times 3 \times 7 \times 2 - 2 \times 3 \times 11 = 858$.

Finally, the generation of an all-pass filter with integer coefficients is direct and quite easy. There are several possibilities to obtain all-pass filters satisfying the five selected criteria. Among all these possibilities, the following choice is proposed:

$$\begin{aligned} a &= [1 + m7, 1 + m11] \\ b &= (1 + m3) \times [1 + m7, 1 - m11], \end{aligned} \quad (6)$$

for the first-order case and

$$\begin{aligned} a &= [1, 1 + m7, 1 + m11] \\ b &= (1 + m3) \times [1, -1 - m7, 1 + m11], \end{aligned} \quad (7)$$

for the second-order case. With this selection, the pole or poles always have a negative real part, and the system can be stable and causal. It is obvious that the selected coefficients for AP filters are integers in the range -33 to 33 . In this way, $3 \times 7 \times 11 \times 2 = 462$ numerically different problems are generated.

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