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A Non-Traditional Numerical Solution to Heat Conduction in a Rectangular Prism*

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This paper presents a new non-traditional approach to computing the numerical solutions of the heat conduction problem of a rectangular prism with and without heat sources or heat sinks subject to Dirichlet and Neumann boundary conditions. The new approach employs a three-dimensional spreadsheet which does not require programming whereas the traditional approach often employs a high-level programming language. The results of the present approach are compared with the analytical results. The advantages and disadvantages are discussed. It is found that the present approach provides some unique and useful features which are not normally achievable by the traditional approach.

INTRODUCTION

IT IS well-known that the heat conduction problem for a rectangular prism with heat sources or heat sinks is governed [1] by the three-dimensional Poisson equation:

$$\nabla^2 T(x,y,z) = f(x,y,z);$$
$$0 \le x \le L,\ 0 \le y \le H,\ 0 \le z \le D \qquad (1)$$

where $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$; $T(x,y,z)$ is the temperature distribution; $f(x,y,z)$ is the source term; L, H and D are the dimensions of the prism; x, y and z are the spatial coordinates.

Analytical solution of the governing equation is possible only for very simple boundary conditions. Traditionally, when the governing equation is to be solved numerically, one usually develops a computer program using a high-level language such as FORTRAN which must be tested and debugged before results can be obtained. Due to the three-dimensionality of the governing equation, these programs invariably use a three-dimensional array to store the solution, regardless of the numerical methods on which they are based. Therefore, the outputs of these programs are difficult to visualise, as they do not contain explicit geometrical information. These programs normally do not have a built-in graphical interface to generate results in graphical form. As such, the graphing is done separately in which either special graphing routines are developed or, as is more often done, the numerical results are exported to an external graphics package for graphing. Careful rearranging and formatting of the numerical results are required before they can be used in an external graphics package. Instead of developing a computer program, some commercially available computational package or scientific computation libraries can also be used. These packages or libraries, together with a suitable graphics post-possessor, can generate the results in impressive graphical forms. However, programming in dedicated languages of these packages is still unavoidable in using them.

By using the spreadsheet, on the other hand, offers some unique features which the traditional approach does not provide. This is even more so for the present case because most recent spreadsheet packages offer a host of advanced three-dimensional features well suited for applications in three-dimensional problems. The cell structure of the spreadsheet naturally provides the physical geometrical locations of the grid points, thus making interpretation of the numerical results easier. The spreadsheet approach is also self-contained in the sense that all necessary tasks can be performed within it including result graphing due to its built-in graphics capability. The spreadsheet is also easier to learn and to use as compared to specialised packages and no programming is necessary.

The applications of spreadsheet in engineering computation is not new. A variety of scientific and engineering problems have been solved by spreadsheets [2–15]. The educational applications ranging from problem simulations [2, 3, 5, 9–12, 15] to course administration [6] have been reported. However, all these applications pertained to at most two-dimensional space and they employed two-dimensional spreadsheets. Recently, Lam [9–11] utilised the spreadsheet to numerically solve the three basic types of second-order partial differential equations in two independent variables using a variety of finite-difference methods and the method of characteristics. For the elliptical equation, only the Laplace equation was solved. Due to the proven educational advantages of these

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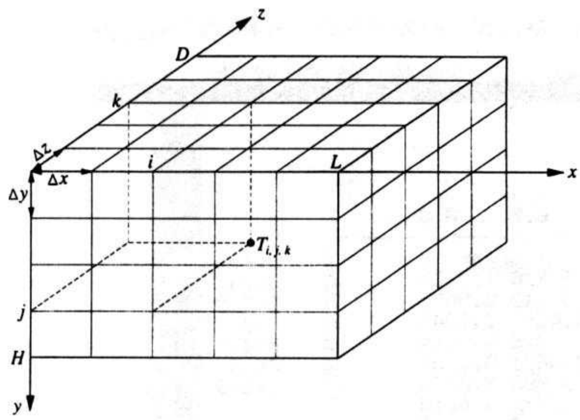


Fig. 1. Discretisation of the rectangular prism.

spreadsheets, they were incorporated into a textbook [1]. Again, only two-dimensional spreadsheets were used for those two-dimensional problems. This paper explores the use of the advanced three-dimensional spreadsheet in solving the three-dimensional heat conduction problem of a rectangular prism with and without heat sources or heat sinks subject to Dirichlet and Neumann boundary conditions. For clear illustration purpose, simple forms of heat sources and boundary conditions are considered.

NUMERICAL PROCEDURE

The governing equation (1) is an elliptic equation. To solve it by a finite-difference method, the prism is discretised by a rectangular grid system with constant grid sizes Δx, Δy and Δz as shown in Fig. 1.

Using central differences [1] to replace the derivative terms in equation (1) and rearranging the resulting equation yields:

$$T_{i,j,k} = \frac{T_{i+1,j,k} + T_{i-1,j,k} + (\Delta x/\Delta y)^2 (T_{i,j+1,k} + T_{i,j-1,k}) + (\Delta x/\Delta z)^2 (T_{i,j,k+1} + T_{i,j,k-1}) - (\Delta x)^2 f_{i,j,k}}{2[1 + (\Delta x/\Delta y)^2 + (\Delta x/\Delta z)^2]} \quad (2)$$

where the subscripts denote the positions of the grid points as depicted in Fig. 1.

The difference equation (2) relates the temperature T at an interior grid point to the values at its six adjacent grid points and the corresponding source term. When it is used to obtain the numerical solution of equation (1), the three-dimensional spreadsheet has an obvious advantage that the spreadsheet structure closely resembles the rectangular grid system shown in Fig. 1. In the present work, the popular three-dimensional spreadsheet package Lotus 1-2-3 Release 3.4a is used.

Figure 2 shows the structure of a three-dimensional spreadsheet in which the location of a spreadsheet cell is defined when its column-wise, row-wise and sheet-wise locations are known. A typical cell on column D and row 5 of sheet B (i.e. at B:D5 in 1-2-3's terminology [16]) is highlighted in Fig. 2 as an example. Therefore, if the column-wise, row-wise and sheet-wise directions represent the x-, y- and z-axis respectively, each cell can be regarded as a grid point in the three-dimensional rectangular grid system. This three-dimensional geometrical information embedded in the three-dimensional spreadsheet structure makes it extremely convenient to assign equation (2) to the cells of a spreadsheet that correspond to the interior grid points of the prism. In fact, the

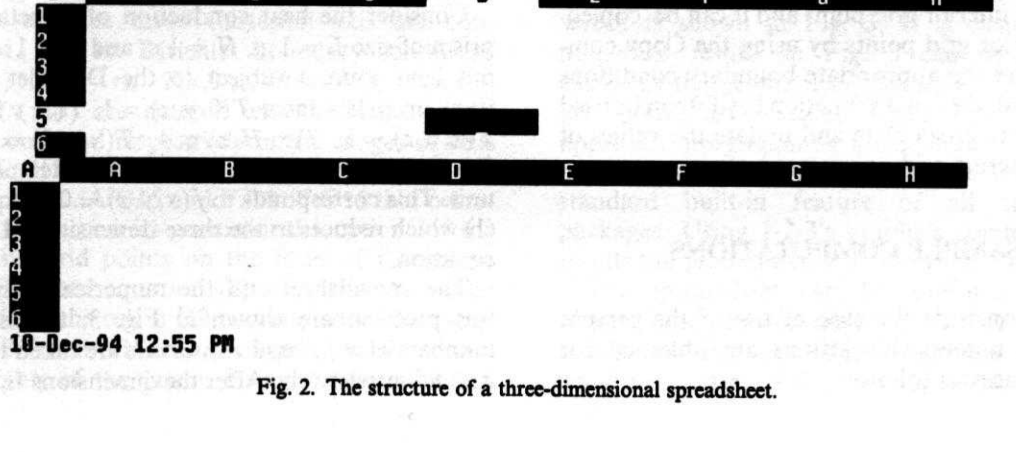
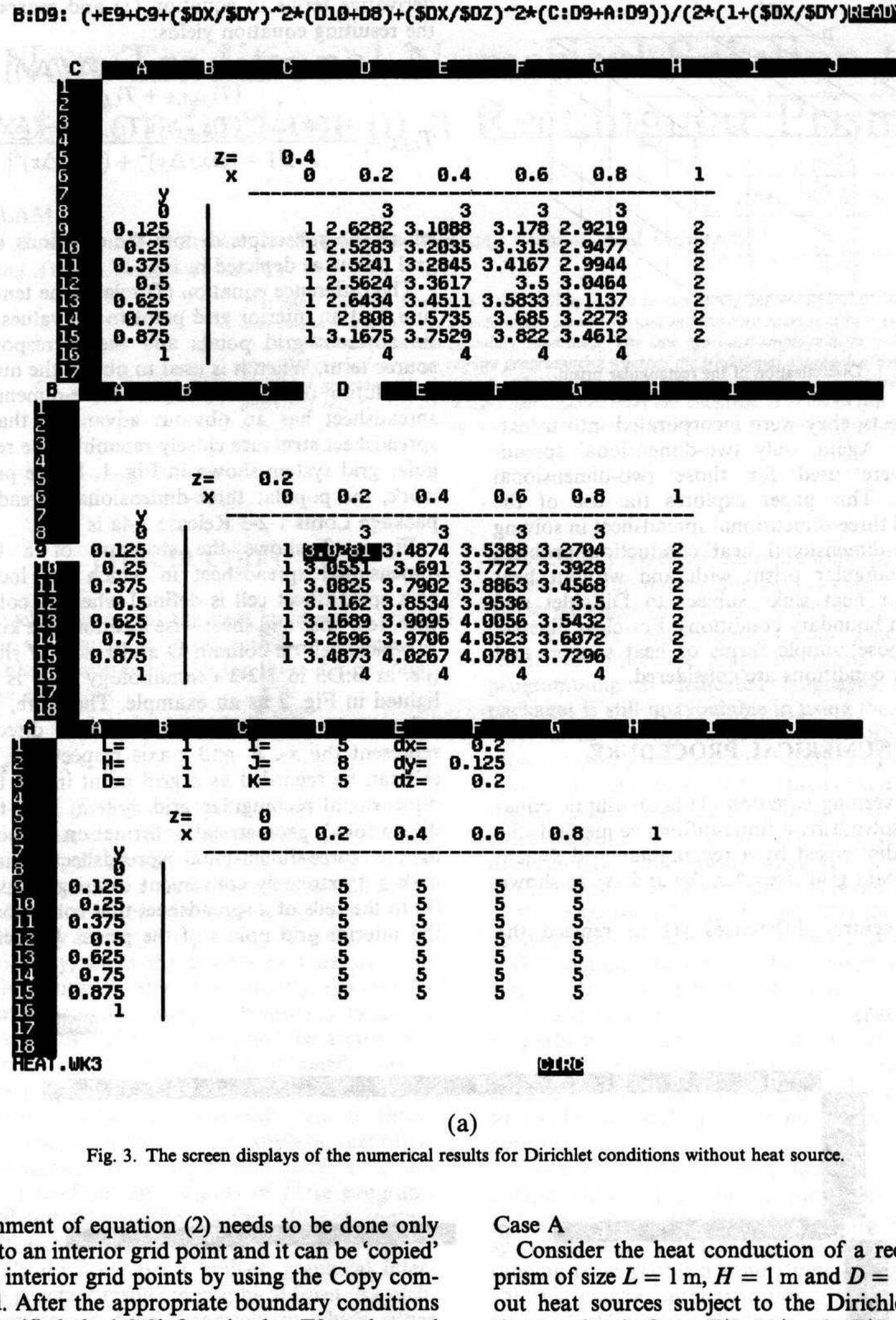


Fig. 2. The structure of a three-dimensional spreadsheet.



B:D9: (+E9+C9+($DX/$DY)^2*(D10+D8)+($DX/$DZ)^2*(C:D9+A:D9))/(2*(1+($DX/$DY) READY

Sheet C: z= 0.4

y \ x	0	0.2	0.4	0.6	0.8	1
0		3	3	3	3	
0.125	1	2.6282	3.1088	3.178	2.9219	2
0.25	1	2.5283	3.2035	3.315	2.9477	2
0.375	1	2.5241	3.2845	3.4167	2.9944	2
0.5	1	2.5624	3.3617	3.5	3.0464	2
0.625	1	2.6434	3.4511	3.5833	3.1137	2
0.75	1	2.808	3.5735	3.685	3.2273	2
0.875	1	3.1675	3.7529	3.822	3.4612	2
1		4	4	4	4	

Sheet B: z= 0.2

y \ x	0	0.2	0.4	0.6	0.8	1
0		3	3	3	3	
0.125	1	3.0481	3.4874	3.5388	3.2704	2
0.25	1	3.0551	3.691	3.7727	3.3928	2
0.375	1	3.0825	3.7902	3.8863	3.4568	2
0.5	1	3.1162	3.8534	3.9536	3.5	2
0.625	1	3.1689	3.9095	4.0056	3.5432	2
0.75	1	3.2696	3.9706	4.0523	3.6072	2
0.875	1	3.4873	4.0267	4.0781	3.7296	2
1		4	4	4	4	

Sheet A:

L=	1	I=	5	dx=	0.2
H=	1	J=	8	dy=	0.125
D=	1	K=	5	dz=	0.2

z= 0

y \ x	0	0.2	0.4	0.6	0.8	1
0						
0.125		5	5	5	5	
0.25		5	5	5	5	
0.375		5	5	5	5	
0.5		5	5	5	5	
0.625		5	5	5	5	
0.75		5	5	5	5	
0.875		5	5	5	5	
1						

HEAT.WK3 CIRC

(a)

Fig. 3. The screen displays of the numerical results for Dirichlet conditions without heat source.

assignment of equation (2) needs to be done only once to an interior grid point and it can be 'copied' to all interior grid points by using the Copy command. After the appropriate boundary conditions are specified, the 1-2-3's function key F9 can be used repeatedly to recalculate and update the values of T at all interior grid points until convergence.

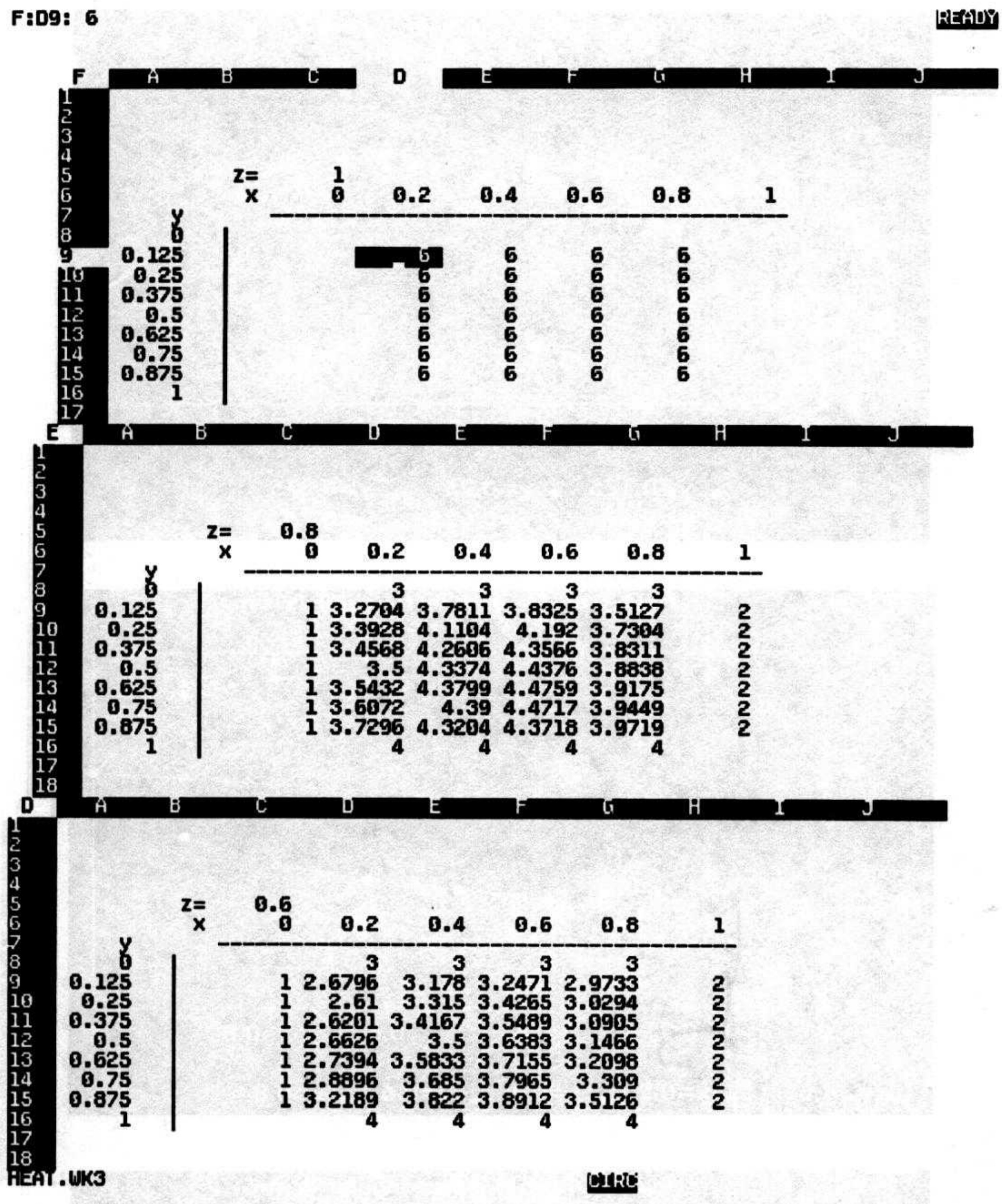
SAMPLE COMPUTATIONS

To demonstrate the ease of use of the present approach, numerical solutions are obtained for different cases as follows:

Case A

Consider the heat conduction of a rectangular prism of size $L = 1$ m, $H = 1$ m and $D = 1$ m without heat sources subject to the Dirichlet conditions on its six faces $T(0, y, z) = 1$, $T(L, y, z) = 2$, $T(x, 0, z) = 3$, $T(x, H, z) = 4$, $T(x, y, 0) = 5$ and $T(x, y, D) = 6$ in an appropriate temperature unit. This corresponds to $f(x, y, z) = 0$ in equation (1) which reduces to the three-dimensional Laplace equation.

The spreadsheet and the numerical results for this problem are shown in Fig. 3 in which the numbers of x-, y- and z-intervals are taken to be 5, 8 and 5 respectively. After the dimensions L, H and

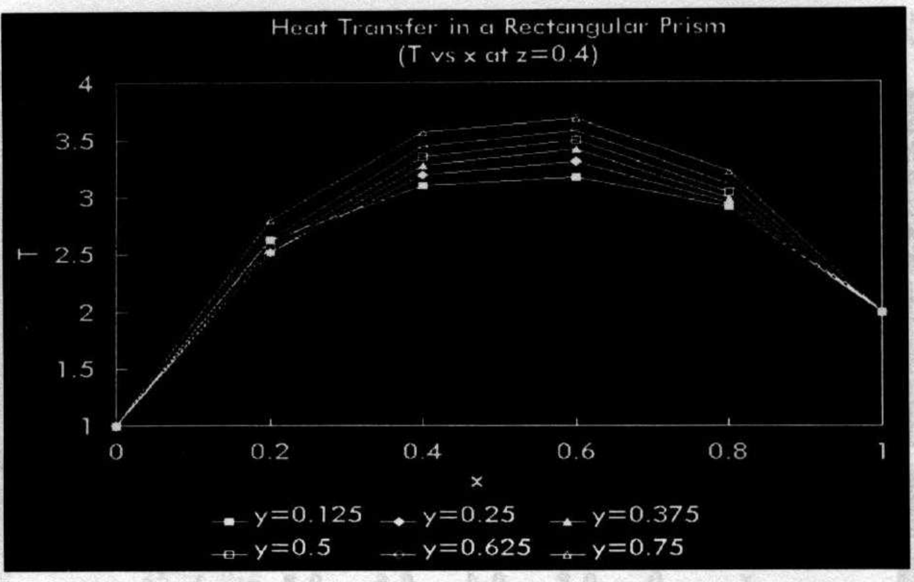


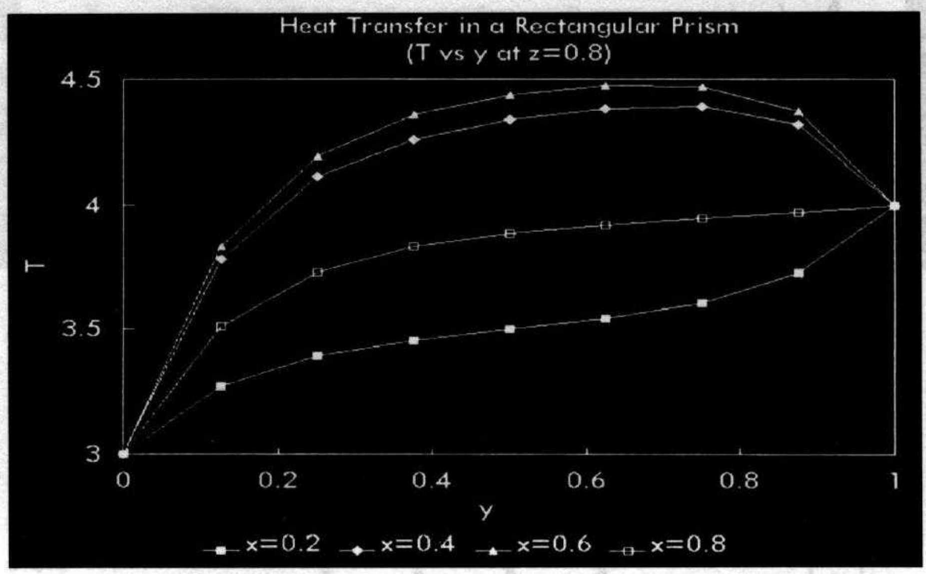
(b)

Fig. 3. (Continued).

D and the number of intervals are specified in the ranges of cells A:B1..A:B3 and A:D1..A:D3 respectively, the three Cartesian axis are constructed. The six Dirichlet boundary conditions $T(0,y,z)$, $T(L,y,z)$, $T(x,0,z)$, $T(x,H,z)$, $T(x,y,0)$ and $T(x,y,D)$ are entered with the help of the Copy command respectively in the ranges of cells B:C9..E:C15, B:H9..E:H15, B:D8..E:G8, B:D16..E:G16, A:D9..A:G15 and F:D9..F:G15 that correspond to the correct geometrical locations of the respective grid points on the faces of the prism. The difference equation (2) with $f = 0$ is then specified in the cell B:D9 and copied to the range of cell B:D9..E:G15 that correspond to all interior grid points of the prism. Using the function key F9, the solution is iterated and the solution converged to four decimal places as displayed on the screen is shown in Fig. 3. It is clear that the numerical results in Fig. 3 can be visualised easily, as the geometrical locations of all cells or grid points are revealed. Unlike the traditional approach, programming is not required and graphics capability is readily available since it is a standard built-in feature of all spreadsheet packages. Using 1-2-3's graphics commands, the results are plotted easily as shown in Fig. 4.

The spreadsheet can be modified easily to obtain solutions for different sets of boundary conditions. For example, if the new boundary conditions are $T(0,y,z) = 0$, $T(L,y,z) = 15$,





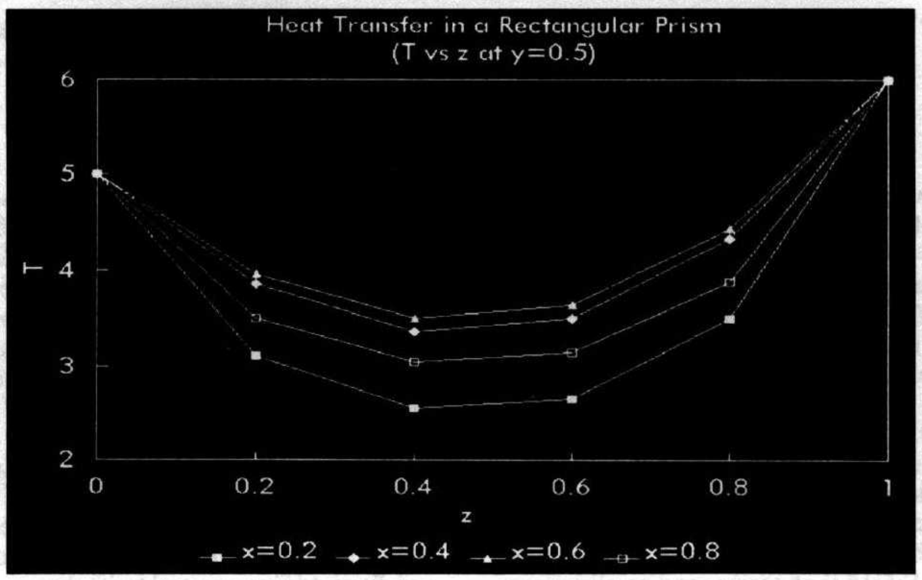
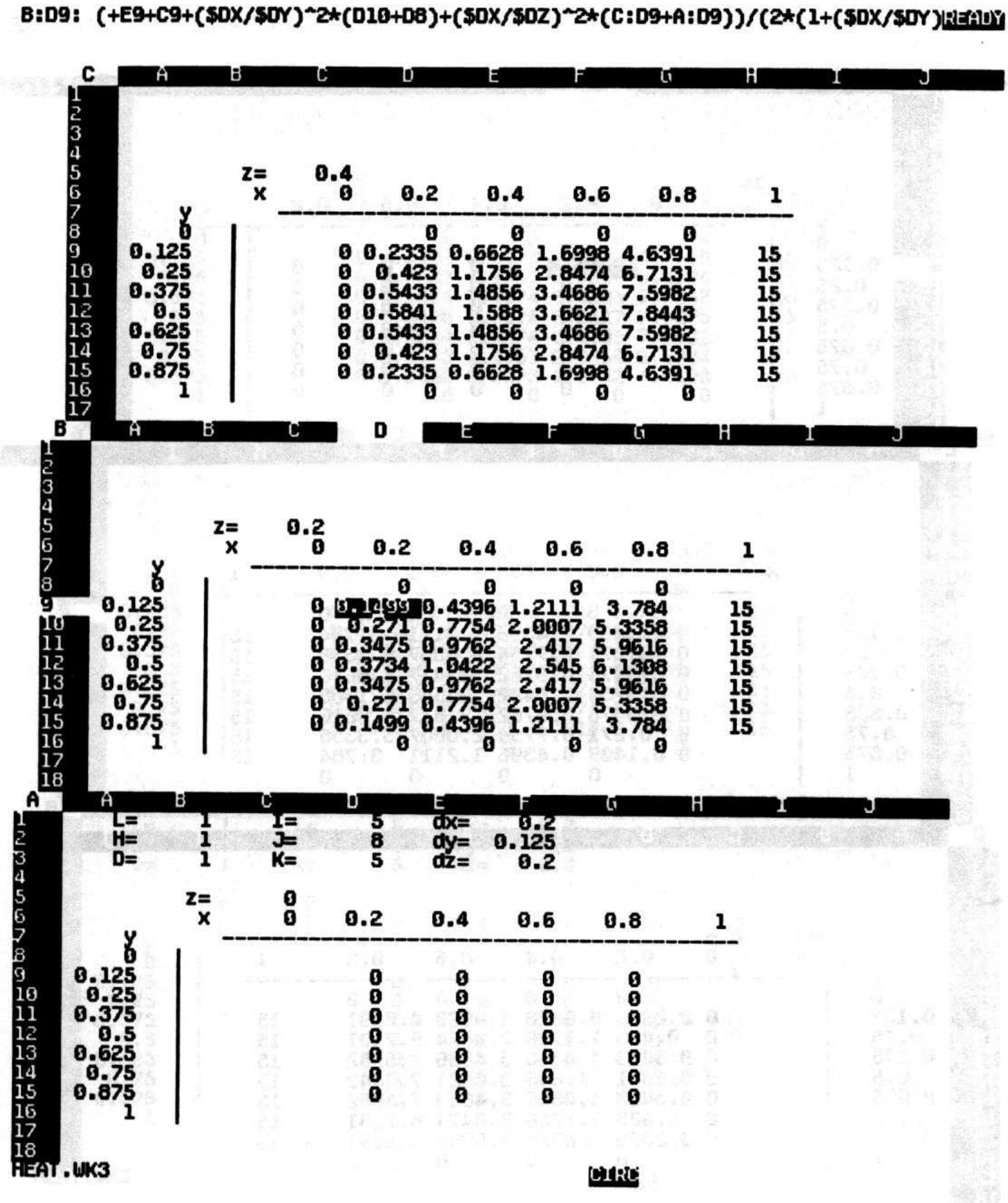


Fig. 4. The graphical displays of the spreadsheet solution.



(a)

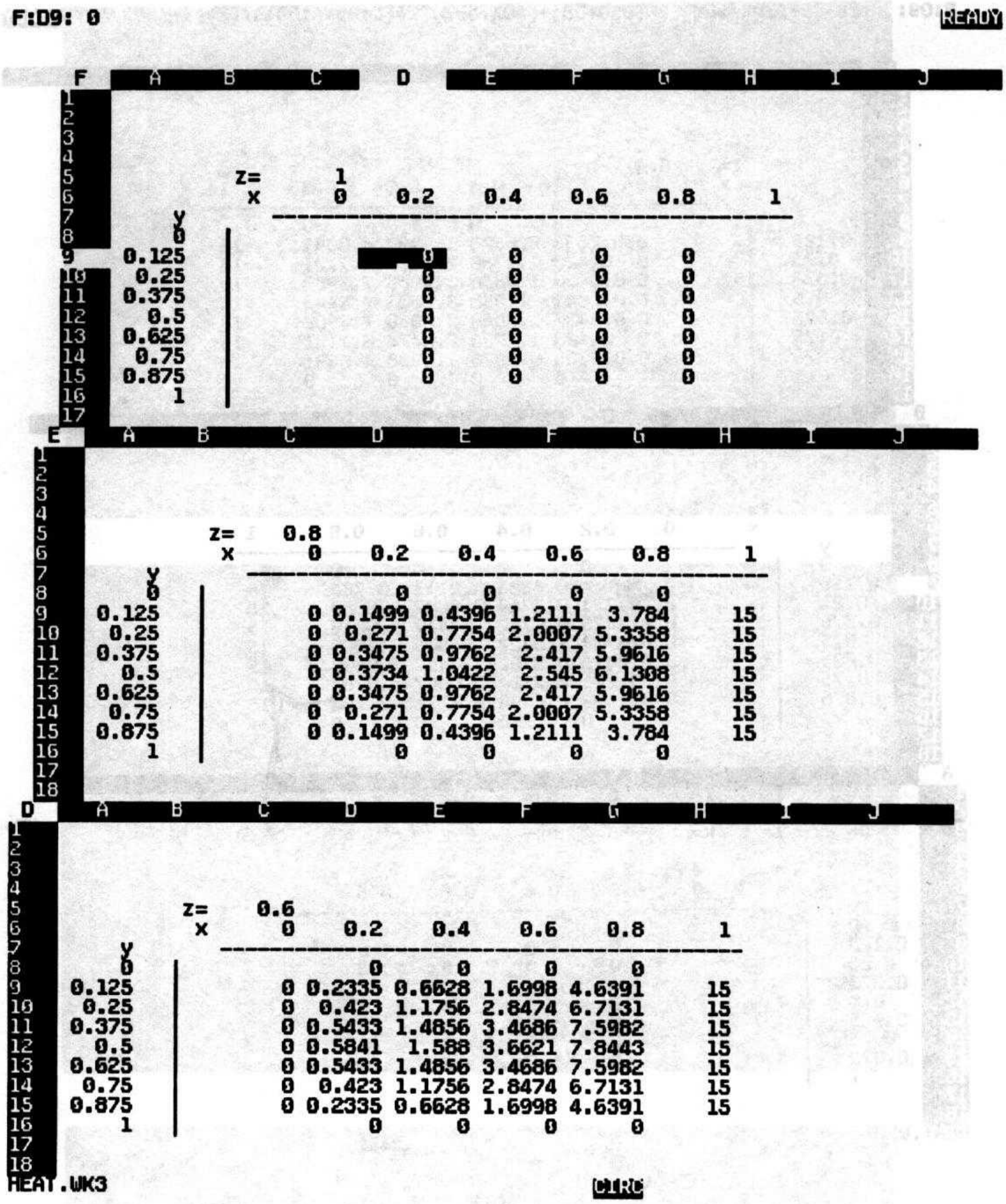
Fig. 5. The screen displays of the results for modified Dirichlet conditions without heat sources.

$T(x,0,z)=0$, $T(x,H,z)=0$, $T(x,y,0)=0$ and $T(x,y,D)=0$, only the corresponding cells which store the boundary conditions identified earlier need to be changed accordingly.

The screen displays of the converged results are shown in Fig. 5. The treatment is the same when the new boundary conditions are not constants. The results for the modified boundary conditions at four preselected grid points are compared in Table 1 with the analytical results obtained by the method of separation of variables. Also included in Table 1 are the results for the same problem when the grid sizes Δx, Δy and Δz are halved. The agreement verifies that the spreadsheet approach is reliable, as expected. This is further supported by the fact that the converged spreadsheet solutions are found identical to the results of the traditional approach using a FORTRAN program with the successive over-relaxation method.

Case B

When heat sources are present $f(x,y,z) \neq 0$ and the Poisson equation (1) must be solved. For illustration, $f(x,y,z)$ is taken to be 1 and the spreadsheet is modified accordingly. This is done easily by editing the difference equation in any one interior grid point following equation (2) with $f=1$ and copying it to all interior grid points. The



z= 1

y \ x	0	0.2	0.4	0.6	0.8	1
0						
0.125		0	0	0	0	
0.25		0	0	0	0	
0.375		0	0	0	0	
0.5		0	0	0	0	
0.625		0	0	0	0	
0.75		0	0	0	0	
0.875		0	0	0	0	
1						

z= 0.8

y \ x	0	0.2	0.4	0.6	0.8	1
0		0	0	0	0	
0.125	0	0.1499	0.4396	1.2111	3.784	15
0.25	0	0.271	0.7754	2.0007	5.3358	15
0.375	0	0.3475	0.9762	2.417	5.9616	15
0.5	0	0.3734	1.0422	2.545	6.1308	15
0.625	0	0.3475	0.9762	2.417	5.9616	15
0.75	0	0.271	0.7754	2.0007	5.3358	15
0.875	0	0.1499	0.4396	1.2111	3.784	15
1		0	0	0	0	

z= 0.6

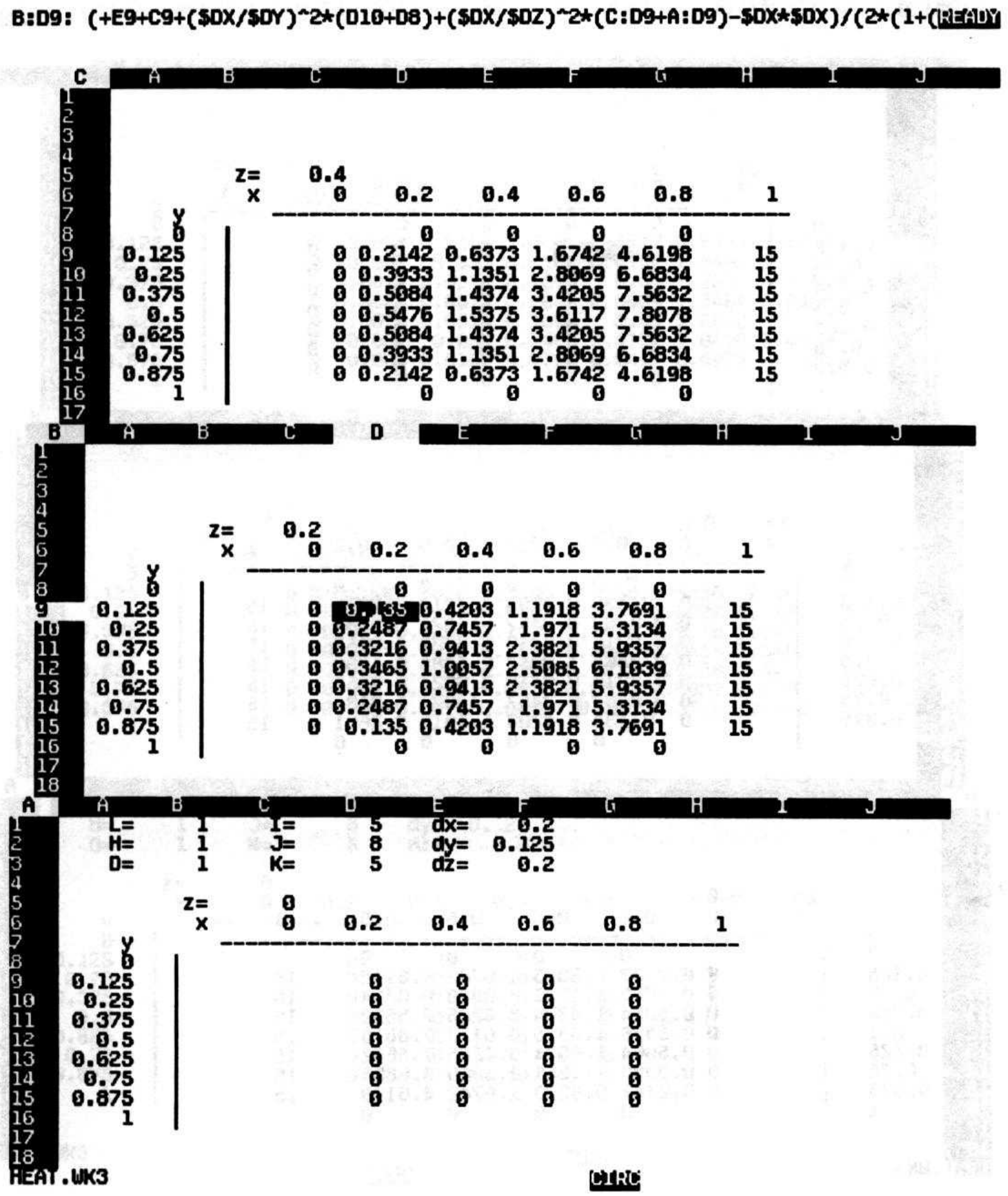
y \ x	0	0.2	0.4	0.6	0.8	1
0		0	0	0	0	
0.125	0	0.2335	0.6628	1.6998	4.6391	15
0.25	0	0.423	1.1756	2.8474	6.7131	15
0.375	0	0.5433	1.4856	3.4686	7.5982	15
0.5	0	0.5841	1.588	3.6621	7.8443	15
0.625	0	0.5433	1.4856	3.4686	7.5982	15
0.75	0	0.423	1.1756	2.8474	6.7131	15
0.875	0	0.2335	0.6628	1.6998	4.6391	15
1		0	0	0	0	

(b)

Fig. 5. (Continued).

Table 1. Comparison of spreadsheet and analytical solutions

	Spreadsheet		
	$\Delta x = 0.2$, $\Delta y = 0.125$, $\Delta x = 0.2$	$\Delta x = 0.1$, $\Delta y = 0.0625$, $\Delta z = 0.1$	Analytical
$T(0.2, 0.125, 0.2)$	0.1499	0.1369	0.1326
$T(0.4, 0.25, 0.4)$	1.1756	1.1280	1.1097
$T(0.6, 0.375, 0.6)$	3.4686	3.4606	3.4591
$T(0.8, 0.5, 0.8)$	6.1308	6.1741	6.1926



(a)

Fig. 6. The screen displays of the results for Dirichlet conditions with heat source.

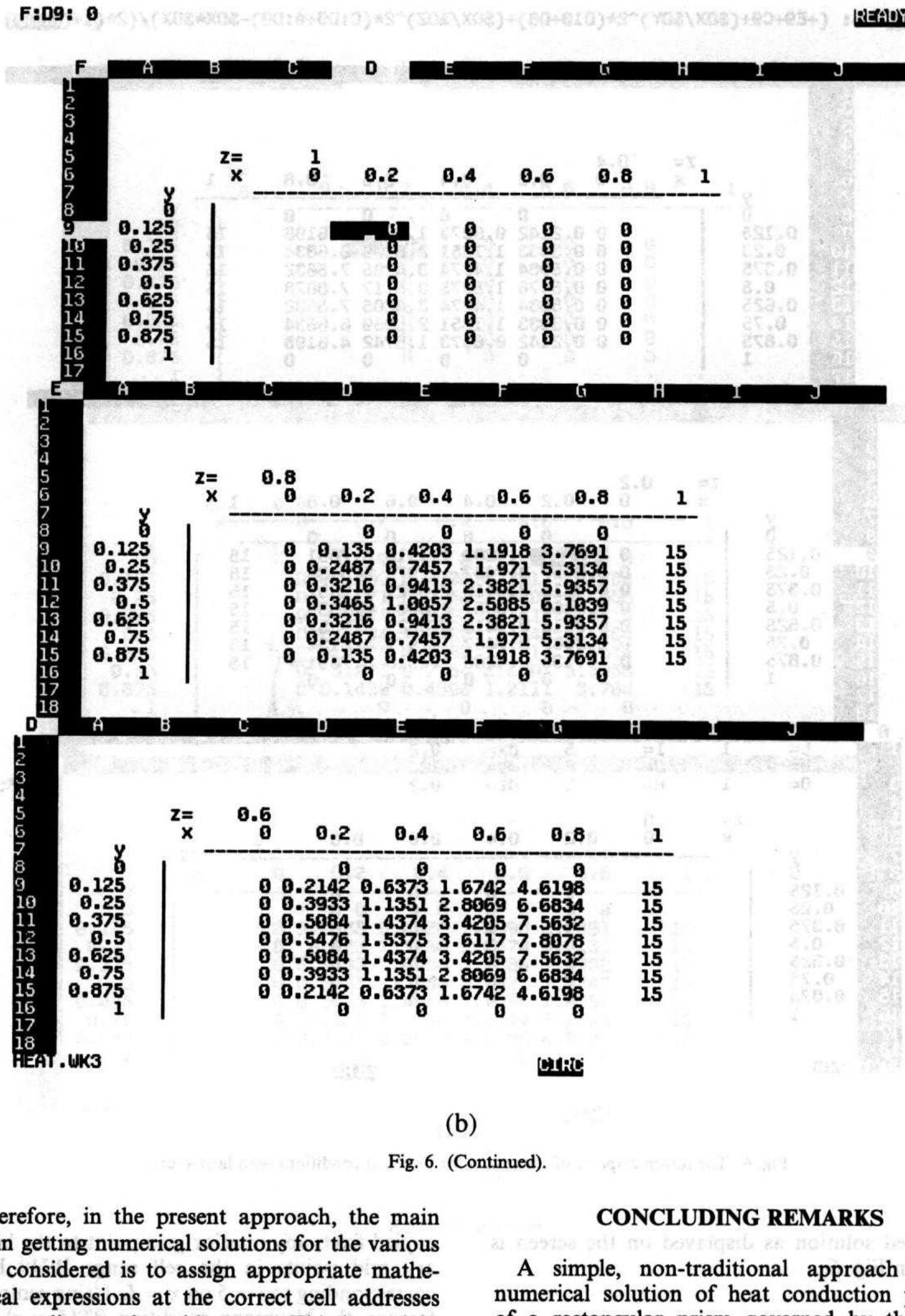
converged solution as displayed on the screen is shown in Fig. 6.

Case C

The Neumann boundary conditions can also be treated easily. Assuming the boundary conditions are $T(0,y,z)=0$, $\partial T(L,y,z)/\partial x=5$, $T(x,0,z)=0$, $T(x,H,z)=0$, $T(x,y,0)=0$ and $T(x,y,D)=0$ with heat sources such that $f(x,y,z)=1$. Here a Neumann condition is specified at $x=L$. Using the concept of fictitious grid point [1], the governing equation is applied on the face $x=L$ of the prism and therefore the difference equation (2) is copied from any interior grid point to the boundary grid points in the cell range B:H9..E:H15 corresponding to $i=5$ on $x=L$. Using central difference, the Neumann condition $\partial T(L,y,z)/\partial x \equiv \partial T_{5,j,k}/\partial x = 5$ is written as:

$$T_{6,j,k} = T_{4,j,k} + 5 \times 2\Delta x$$

$T_{6,j,k}$ are the temperatures at the fictitious grid points in the cell range B:I9..E:I15 corresponding to $x = L + \Delta x$ outside the prism. This equation is entered at one fictitious grid point and copied to all fictitious grid points. Again, iteration is carried out by using the F9 function key and the converged solution is shown in Fig. 7.



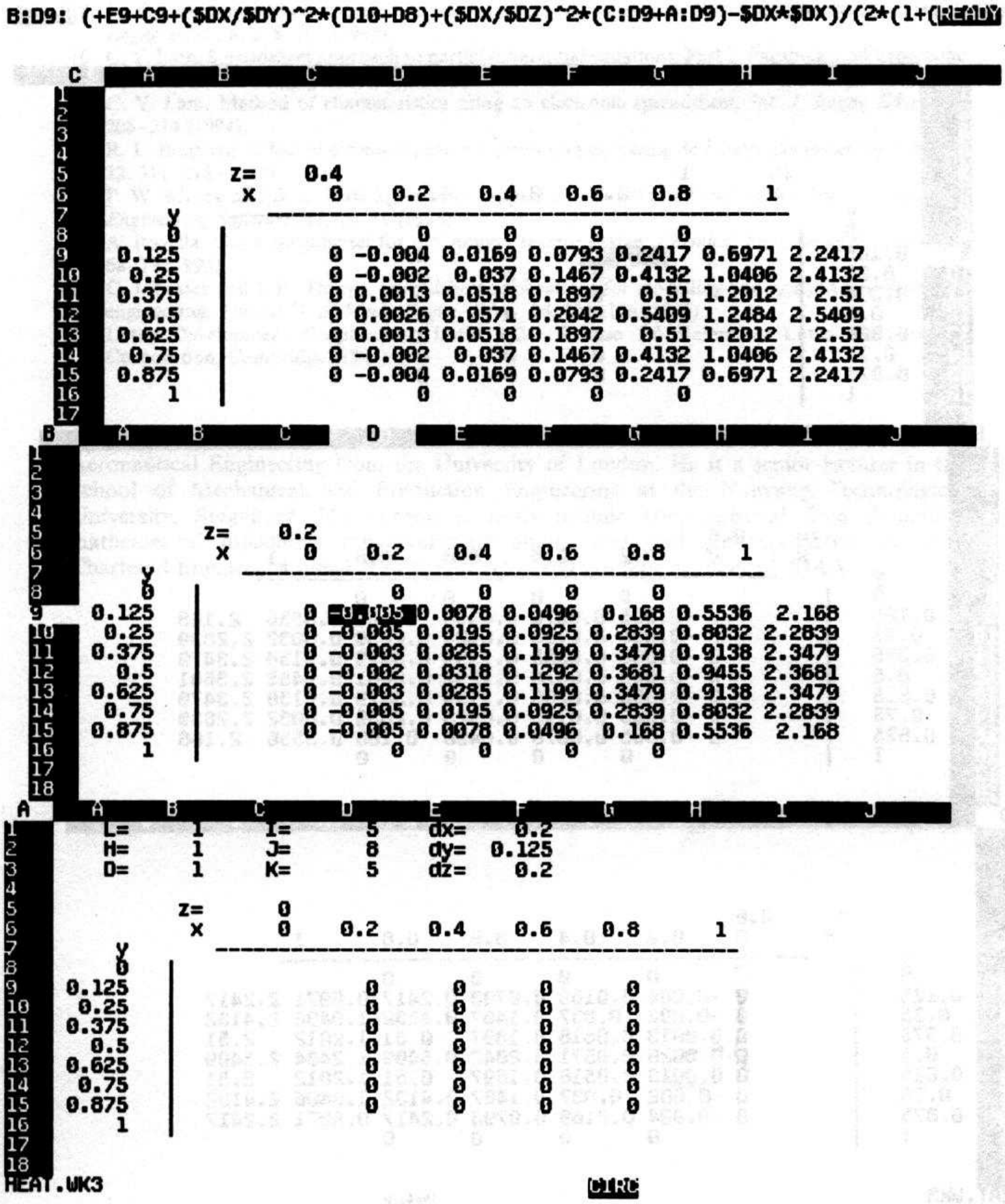
(b)

Fig. 6. (Continued).

Therefore, in the present approach, the main task in getting numerical solutions for the various cases considered is to assign appropriate mathematical expressions at the correct cell addresses corresponding to the grid system. This is straight forward because the cell structure resembles the three-dimensional grid system. The treatment of Neumann boundary conditions, which is much more troublesome as compared to Dirichlet conditions when the traditional approach is adopted, is basically as straightforward as that of the Dirichlet condition with the additional assignment of mathematical expressions at the fictitious grid points.

CONCLUDING REMARKS

A simple, non-traditional approach to the numerical solution of heat conduction problem of a rectangular prism governed by the three-dimensional Poisson equation is presented. While the Dirichlet boundary conditions can be treated in a straightforward manner, the Neumann condition can also be easily imposed using the concept of fictitious grid points which offers second-order accurate results. The present approach utilises the built-in features of a three-dimensional spreadsheet. It is only required to set up the computational grid system, to specify the boundary conditions and the relations between the cell



y \ x	0	0.2	0.4	0.6	0.8	1	
0		0	0	0	0		
0.125	0	-0.004	0.0169	0.0793	0.2417	0.6971	2.2417
0.25	0	-0.002	0.037	0.1467	0.4132	1.0406	2.4132
0.375	0	0.0013	0.0518	0.1897	0.51	1.2012	2.51
0.5	0	0.0026	0.0571	0.2042	0.5409	1.2484	2.5409
0.625	0	0.0013	0.0518	0.1897	0.51	1.2012	2.51
0.75	0	-0.002	0.037	0.1467	0.4132	1.0406	2.4132
0.875	0	-0.004	0.0169	0.0793	0.2417	0.6971	2.2417
1		0	0	0	0		

y \ x	0	0.2	0.4	0.6	0.8	1	
0		0	0	0	0		
0.125	0	-0.005	0.0078	0.0496	0.168	0.5536	2.168
0.25	0	-0.005	0.0195	0.0925	0.2839	0.8032	2.2839
0.375	0	-0.003	0.0285	0.1199	0.3479	0.9138	2.3479
0.5	0	-0.002	0.0318	0.1292	0.3681	0.9455	2.3681
0.625	0	-0.003	0.0285	0.1199	0.3479	0.9138	2.3479
0.75	0	-0.005	0.0195	0.0925	0.2839	0.8032	2.2839
0.875	0	-0.005	0.0078	0.0496	0.168	0.5536	2.168
1		0	0	0	0		

y \ x	0	0.2	0.4	0.6	0.8	1
0						
0.125		0	0	0	0	
0.25		0	0	0	0	
0.375		0	0	0	0	
0.5		0	0	0	0	
0.625		0	0	0	0	
0.75		0	0	0	0	
0.875		0	0	0	0	
1						

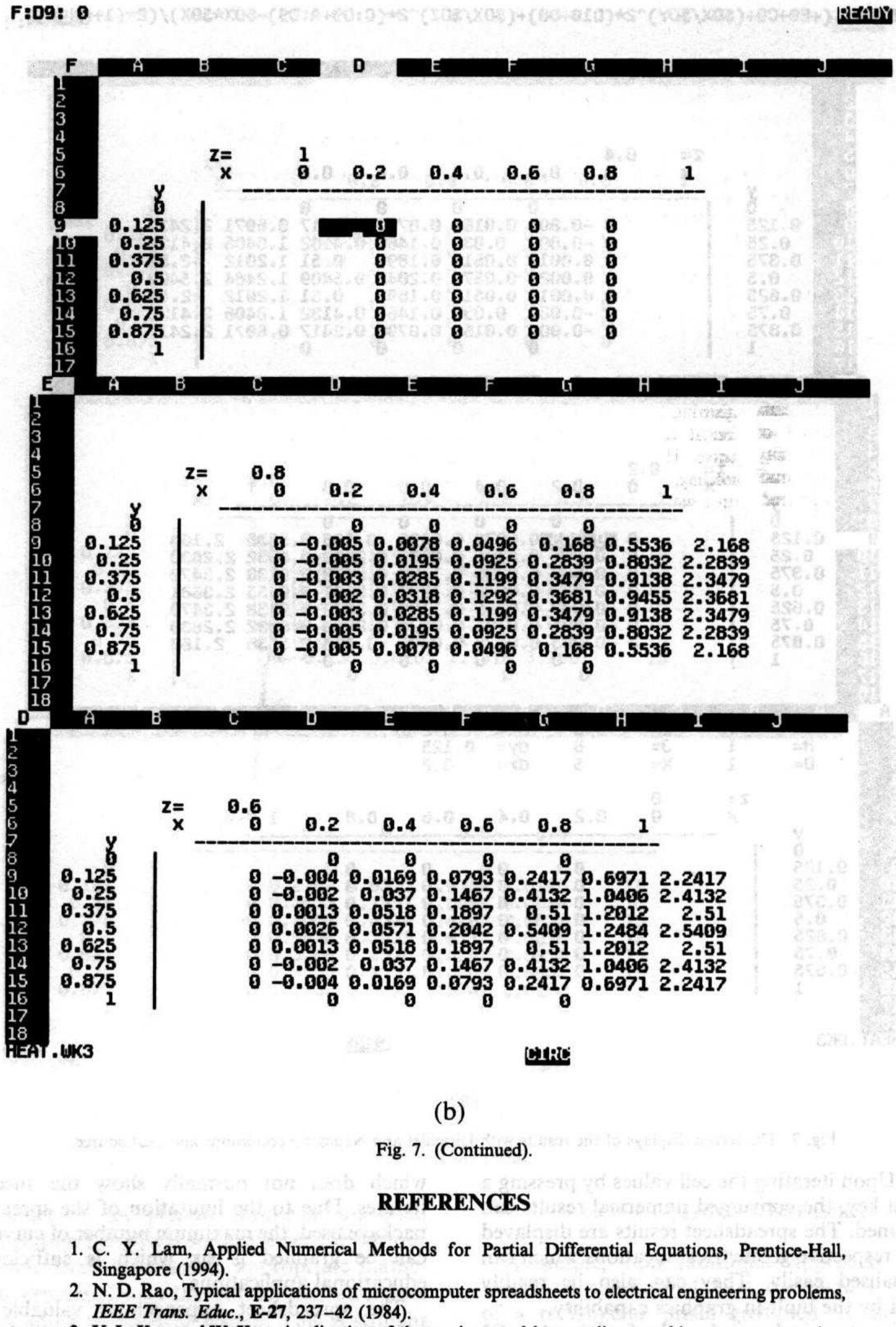
(a)

Fig. 7. The screen displays of the results with Dirichlet and Neumann conditions and heat source.

values. Upon iterating the cell values by pressing a function key, the converged numerical results can be obtained. The spreadsheet results are displayed at their respective geometrical locations which can be visualised easily. They can also be readily graphed by the built-in graphics capability.

One iteration is done when the function key F9 is pressed once. This can in some cases be a little troublesome especially when the number of grid points is large for which convergence is slow. However, this allows the convergence characteristics of the solution to be studied as the successive iterates can be observed after each iteration. For educational purposes, this can be another advantage over the traditional approach of writing a computer program using a high-level language which does not normally show the successive iterates. Due to the limitation of the spreadsheet package used, the maximum number of curves that can be graphed is six which is sufficient for educational applications.

The spreadsheet approach is valuable since programming, compilation and debugging are not required which can often be frustrating. Spreadsheet packages are easily accessible to engineering students. Most students nowadays also have spreadsheet packages installed in their home personal computers and may have adequate spreadsheet knowledge to carry out the numerical computation. The spreadsheet approach thus provides a simpler, faster and yet powerful alternative to the traditional approach.



(b)

Fig. 7. (Continued).

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