Int. J. Engng Ed. Vol. 12, No. 5, p. 382–388, 1996
Printed in Great Britain.
0949-149X/91 $3.00+0.00

Stability Analysis and Minimum Phase Testing of Causal Systems Using Spreadsheets*

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This paper introduces a spreadsheet algorithm for determining the stability of an n*-th order causal system using Jury's method. The difficulty of this method for higher-order polynomials and its use to check for minimum-phase conditions are discussed. The number of poles and/or zeros inside and outside the z-plane unit circle is also determined from the worksheet. Furthermore, other stability tests are discussed and several examples are given as illustration.*

EDUCATIONAL SUMMARY

1. The paper discusses materials for a course in discrete systems.
2. Students of the electrical engineering department are taught in this course.
3. The course is for senior year students.
4. The course is presented via computer software.
5. The material is presented in a regular course.
6. Three hours are required to cover the material.
7. Two hours of student homework or revision hours are required for the material.
8. The novel aspects presented in this paper are the use of an already available and widely used software tool instead of using a special engineering software tool.
9. The standard text recommended in the course, in addition to author's notes is A. Antonio, *Digital Filters: Analysis and Design*, McGraw-Hill (1979).
10. The material is covered in the text.

1. INTRODUCTION

THE USE of spreadsheet programs in solving engineering problems has proven to be an important tool for PC users who are not expert in computer programming. Such software provides an easy way for input, and, accordingly, allows the user to monitor sudden changes in the solution. For these reasons, a spreadsheet was used as per the examples, to study the stability of linear control systems using the Routh–Hurwitz criterion [1], solve some linear optimization problems [2], simulate logical networks [3], design digital filters [4], and for high-level mixed-mode simulation [5].

In this paper, Lotus 1-2-3 was used to study the stability of an n-th order causal filter using Jury's method. This method is extended to determine whether the system under consideration is a minimum-phase system. Special cases have been taken into consideration and results are readily available. The difficult part of the problem is related to the dynamic size of the worksheet table and to the shifting process of the coefficients, which varies with the order of the polynomial. Part 3 includes examples of several case studies. In Part 4, this method is used to test for the minimum-phase condition. This procedure is recommended for its simplicity, flexibility, and the availability of spreadsheets on PCs providing a user-friendly environment.

2. JURY'S STABILITY CRITERION

A digital recursive filter is checked for its stability by determining the location of its poles. A necessary and sufficient condition for stability is that the poles must be located inside the unit circle of the z-plane.

Several analytical and graphical tests are available. Bilinear transformation [6] has been used to map the unit circle into the left half of the s-plane and then the Routh–Hurwitz stability criterion is applied to the corresponding polynomial. This procedure involves a complicated algebraic manipulation and has not been recommended [7]. Stability tests directly applied in the z-plane, on the other hand, are recommended since they are easier to implement and calculate. Some of these

* Accepted 20 April 1996

methods, such as the determinant method and the new stability method [8] which is similar to the Hurwitz or Lienard–Chepart method for the continuous case, require the solution of two algebraic equations of degree $(n-1)$. These criteria discuss the necessary and sufficient conditions for all the poles if the system transfer function is to lie inside the unit circle. Further computation is still required to determine the exact number of poles inside or/and outside the unit circle.

A tabular form of the stability test for causal filters over the unit circle in the z-plane known as Jury's method [9] is considered. This test requires the form of a table where the elements' computation requires the evaluation of a second-order determinant. This method can be easily programmed, as shown in Fig. 1, and is considered to be useful for testing as well as for design procedure. It is based on the coefficients of the polynomial $D(z)$ forming the denominator of the system transfer function $H(z)$. This method is summarized as follows.

Let

$$D(z) = \sum_{i=0}^{n} a_i z^i \tag{1}$$

be the denominator of the filter transfer function $H(z)$. Assuming positive a_n, the coefficients of $D(z)$ are used to form the array below.

Row	Coefficients					
1	a_0	a_1	a_2	$\cdots$	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	$\cdots$	a_1	a_0
3	b_0	b_1	b_2	$\cdots$	b_{n-1}	
4	b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_0	
5	c_0	c_1	c_2	$\cdots$	c_{n-2}	
6	c_{n-2}	c_{n-3}	c_{n-4}	$\cdots$	c_n	
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
$2n-3$	r_0	r_1	r_2			

where

$$b_i = a_0 \cdot a_i - a_n \cdot a_{n-i} \qquad i = 0, 1, \ldots, n-1 \tag{2}$$

$$c_i = b_0 \cdot b_i - b_{n-1} \cdot b_{n-1-i} \quad i = 0, 1, \ldots, n-2 \tag{3}$$

etc.

The d's are also determined similarly to the c's by using the fifth and sixth rows. The procedure continued until $(2n-3)$ rows are obtained. The last row will comprise three elements (r_0, r_1, and r_2).

Jury's method states that the filter characterized by $H(z)$ is stable if and only if the following conditions are satisfied:

(a) $D(1) > 0$
(b) $(-1)^n D(-1) > 0$
(c) $a_n > |a_0|$
$|b_0| > |b_{n-1}|$
$|c_0| > |c_{n-2}|$

The number of roots of $D(z)$ inside the unit circle, if none of the above constraints becomes equal, is given by the number of products P_k, $k = 1, 2, 3, \ldots, n$, which are negative, where

$$P_1 = [|a_0| - a_n]$$

$$P_2 = [|a_0| - a_n][|b_0| - |b_{n-1}]$$

$$\cdots$$

$$P_n = [|a_0| - a_n][|b_0| - |b_{n-1}] \cdots$$
$$[|r_0| - |r_2|][|r_0^2 - r_2^2| - |r_0 r_1 - r_1 r_2|] \tag{4}$$

Figure 1 shows a general flowchart for the different steps involved. More details about the Jury's method is given in [10].

3. SPREADSHEET FORMULA IMPLEMENTATION

The spreadsheet formulas presented herein are for polynomials of degree from 3 to 10. Higher-degree polynomials may be easily accommodated by copying the formulas down the sheet. In order to make the data entry as simple as possible, the spreadsheet was divided into three sections: (i) input section; (ii) calculation section; and (iii) results section.

Input section

The user enters in cell B1 the highest degree of the polynomial. In addition, the user enters 0 into cell B2 and a value of 1 in cell C2. These two values are necessary for the automatic computation of the polynomial degrees $(0 \ldots n)$ which are displayed in cells B2..L2. This gives the user a visual view of the inputted coefficient degree (z^i). The formula for counting and displaying the polynomial degrees is given by:

```
@IF(@ISERR(C2)#OR##NOT#@N(C2..C2),
    " ",@IF(C2 = $B$1," ",C2 + 1)        (5)
```

This formula is implemented at cell D2 and copied into subsequent cells from E2..L2.

Calculation section

In this section, Jury's method is implemented using Lotus 1-2-3 formulas. This is characterized by three formulas. The first formula draws a line separator between different computed coefficients (i.e. between Jury's array row 2 and 3, 4 and 5, etc.). The second computes the reverse order arrangement of the coefficients. The last formula computes the coefficients b_i, c_i, d_i, etc., from a_i, b_i, c_i, etc., respectively.

The formula for drawing a separator line '- - - - -' between groups of similar coefficients is given by:

```
@IF((@CELL("COL",B6..B6)
    <= ($B$1 + (-@CELL("ROW",B6..B6) + 12)/3))
       #AND#(@CELL("ROW",A6..A6)
    <= ($B$1*3-3)),"-----"," ")          (6)
```

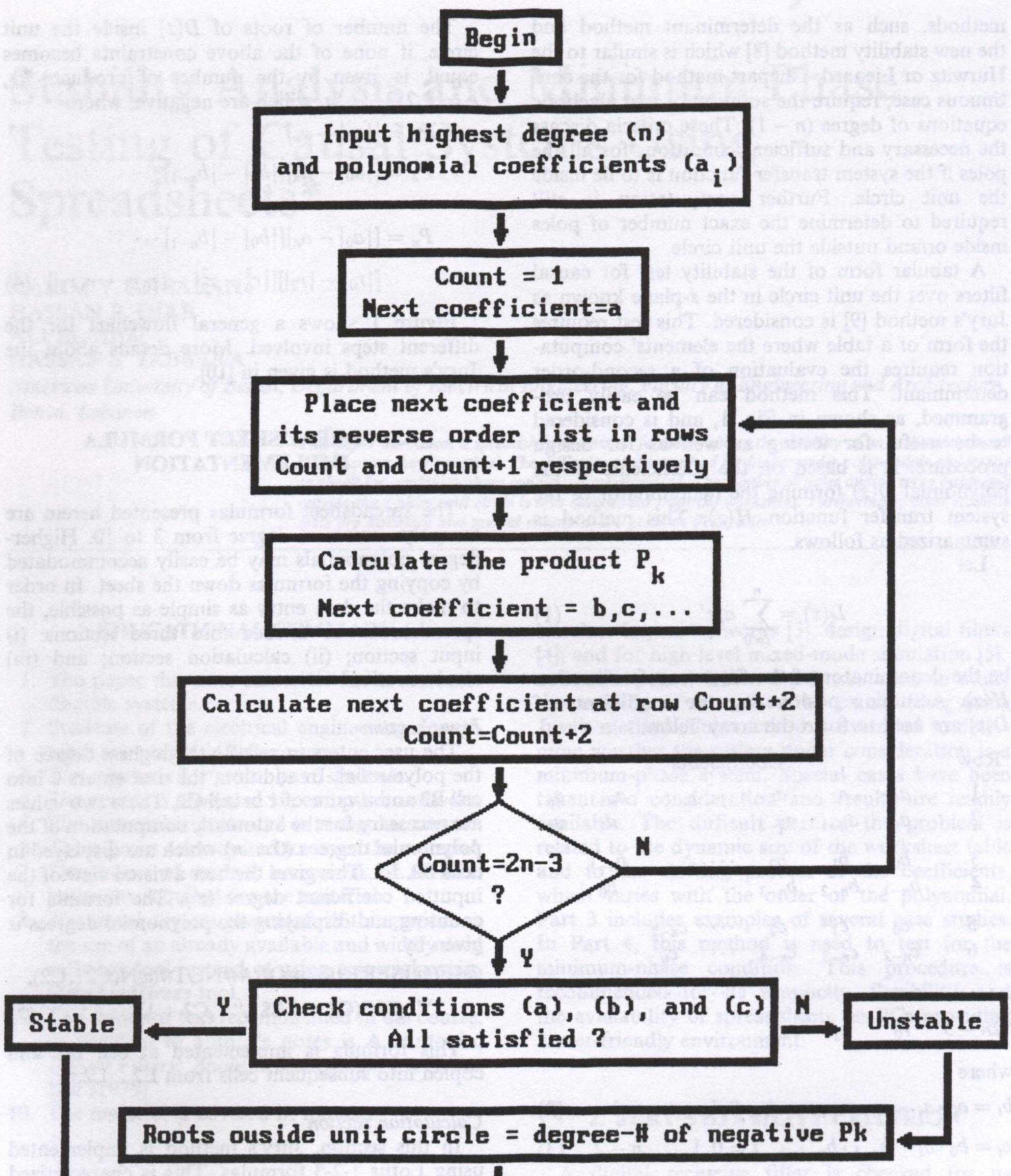

Fig. 1. General flowchart of Jury's method.

This formula is implemented on cell B6 and copied accordingly to cells C6..L6, B9..L9, B12..L12, ..., B27..L27.

The second formula computes the reverse order listing of the coefficients, and is implemented on cell B5 and copied accordingly to C5..L5, B8..L8, ..., B26..L26. This formula is given by:

```
@IF(B6 = "-----",@IF(@ISERR
(@INDEX($B4..$L4,$B$1
+(−@CELL("ROW",B5..B5)+11)/3
−@CELL("COL",B5..B5),0)),"",
@INDEX($B4..$L4,$B$1
+(−@CELL("ROW",B5..B5) + 11/3
−@CELL("COL",B5..B5),0)),"")          (7)
```

The formula computes the reverse order of the coefficients if the cell beneath contains the value '-----'. Thus, the separator line is used as a criteria for computing new coefficients.

The last formula computes the new set of coefficients based on the old ones using the matrix defined by the last section. The formula is implemented on cell B7 and copied accordingly to cells C7..L7, C10..L10, ..., C28..L28, and is given by:

@IF(B6 = "-----", + $B4*@INDEX
($B4..$L4,@CELL("COL",B7..B7) − 2, 0)
−@INDEX($B4..$L4,
B1 + (−@CELL("ROW",B7..B7)
+7/3,0)*@INDEX($B4..$L4,B1
−@CELL("COL",B7..B7)
+(−@CELL("ROW",B7..B7) + 13/3,0)," ") (8)

To determine the number of roots outside the unit circle, the number of products P_k which are negative are computed. This is done by calculating the products P_1 at cell B30 using the following formula:

@ABS(B4) − @INDEX($B4..$L4,B1,0) (9)

The products $P_2 \dots P_{n-1}$ are implemented at cells C30..L30 using the following formula at cell C30 and copying it to D30..L30:

@IF(@CELL("COL",C30..C30)
<= B1,+B30*(@ABS(@INDEX
(B4..L28,0,3*@CELL
("COL",C30..C30) − 6))
− @ABS(@INDEX(B4..L28,
B1-@CELL("COL",C30..C30)
+ 2,3*@CELL("COL",C30..C30) − 6)))" ") (10)

Finally, the product P_n is implemented at cell M30 and is given by:

@INDEX($B30..$L30,B1 − 2,0)*(@ABS(O5^2
−Q5^2) − @ABS(O5*P5 − P5*Q5)) (11)

Cells O5..Q5 are used as temporary storage areas for the coefficients r_0, r_1, and r_2, respectively, which are used to calculate the product P_n. Cell O5 holds the following formula which is copied to cells P5 and Q5:

@INDEX(B4..L28,0,3*B1 − 6) (12)

Results section

The last section implements formulas to check the stability of the system using Jury's criterion. The first condition [condition (a) in Part 2] of the stability analysis is implemented on cell N1 which puts the value 'OK' if the condition is satisfied, or 'FALSE' otherwise.

@IF(@SUM(B4..L4) > 0,"OK","FALSE") (13)

The second stability condition [(b) in Part 2] is evaluated by computing on cells O1..Y1 the following formula:

@IF(@N(B4..B4),+B4*(−1)^B2,0) (14)

and then computing at cell N2 the following formula:

@IF((−1)^B1*@SUM(O1..Y1) > 0,
"OK","FALSE") (15)

The third stability criterion [(*c*) in Part 2] is computed by implementing the following formula at cell N4:

@IF(@ABS(B4)
< @ABS(@INDEX(B4..L4,B1,0)),
"OK","FALSE" (16)

and implementing the following formula at N7 and then copying it to N10, N13, ..., N28:

@IF($B6 = "-----",@IF(@ABS($B7)
> (@ABS(@INDEX($B7,$L7,
B1 − (@CELL("ROW",N7..N7) − 4)/3,0))),
"OK","FALSE")," ") (17)

The stability condition message is displayed at cell N30, which displays the value 'UNSTABLE' if any of the cells (N1, N2, N4, N7, N10, ..., N28) contain 'FALSE', and 'STABLE' otherwise.

The number of roots outside the unit circle is calculated at cell M32, contains the following formula:

+B1 − @SUM(O31..Z31) (18)

This formula computes the number of roots outside the unit circle by subtracting from the degree of the polynomial, the number of negative products P_i. In order to count the number of negative products implemented at cells B30..M30, the following temporary cells (O31..M30) are calculated, which hold a value 1 if the corresponding product is negative, and 0 otherwise:

@IF(@N(B30..B30),@IF(B30 < 0, 1, 0), 0) (19)

This formula is implemented at cell O31 and copied accordingly to cells P31..M31.

Figure 2 shows a sample sheet of Jury's method implemented on Lotus 1-2-3. The system considered is of degree 6, and as shown in the bottom right corner of the sheet, it is stable since all stability conditions are satisfied. In addition, the number of poles outside the unit circle is displayed below and is equal to 0. Figure 3

```
Highest Degree=        6                                                  [a]   OK
Degree:                0     1     2     3     4     5     6              [b]   OK
==========================================================================
Enter Coefficients:    1     1     2     3     4     5     6              [c]   OK
                       6     5     4     3     2     1     1
-------------------------------------------------------------
                     -35   -29   -22   -15    -8    -1     0                    OK
                      -1    -8   -15   -22   -29   -35
-------------------------------------------------------
                    1224  1007   755   503   251     0                          OK
                     251   503   755  1007  1224
-------------------------------------------------
                   1E+06 1E+06 734615 362915     0                              OK
                   4E+05 7E+05 1E+06  1E+06
-------------------------------------------
                   2E+12 1E+12 7E+11      0                                     OK

=================================================================================  ========
P(i=1..n)             -5  -170 -2E+05 -2E+11 -2E+23                      -4E+47  STABLE
                                                                                  ========
                                                                              0  ROOTS OUT
                                                                                  ========
```

Fig. 2. A sample worksheet of Jury's method for a stable system of degree 6.

shows an unstable system of degree 8 with four roots outside the unit circle.

4. APPLICATION TO MINIMUM-PHASE SYSTEMS

In this section, in the term 'minimum-phase system' is used to denote a system whose frequency response is minimum phase. This condition requires that there be no poles or zeros of the system transfer function $H(z)$ outside the unit circle. In order to work out this problem using this sheet, the test is repeated twice, first for the coefficients of the polynomial representing the denominator of $H(z)$, and second for the coefficients of the polynomial representing the numerator of $H(z)$. If both tests resulted in a stable response, the system is then stable and minimum phase. Moreover, if the first test is stable, and the second is not, then the system is stable but non-minimum phase.

This study has been applied to the transfer function of the sixth-order system given by:

$$H(z) = \frac{\frac{1}{12} + \frac{1}{4}z^2 + \frac{1}{3}z^4 + z^6}{\frac{1}{18} + \frac{1}{9}z^2 + \frac{1}{2}z^4 + z^6} \tag{20}$$

The result for stability and minimum-phase

```
Highest Degree=        8                                                    [a]   OK
Degree:                0      1      2     3     4     5      6     7    8  [b]   OK
============================================================================
Enter Coefficients:    1      1      2     3     4    -1      3   0.1 0.31  [c]   FALSE
                    0.31    0.1      3    -1     4     3      2     1    1
---------------------------------------------------------------------------
                  0.9039  0.969   1.07  3.31  2.76  -1.9   2.38  -0.2    0        OK
                   -0.21   2.38  -1.93  2.76  3.31  1.07  0.969   0.9
---------------------------------------------------------------------
                  0.7729  1.376  0.562  3.57  3.19  -1.5  2.355     0             FALSE
                  2.3548  -1.52   3.19  3.57  0.56  1.38  0.773
---------------------------------------------------------------
                  -4.948  4.642  -7.08  -5.6  1.14  -4.4      0                   OK
                  -4.414  1.142  -5.65  -7.1  4.64  -4.9
---------------------------------------------------------
                  4.9934  -17.9  10.08  -3.3  14.8     0                          FALSE
                  14.839  -3.29  10.08   -18  4.99
---------------------------------------------------
                  -195.3  -40.7  -99.2   250     0                                FALSE
                  249.55  -99.2  -40.7  -195
---------------------------------------------
                  -24152  32706  29529     0                                      FALSE

=================================================================================  ========
P(i=1..n)           0.69  0.479  -0.76  -0.4  3.98  -216  1E+06          -2E+15  UNSTABLE
                                                                                  ========
                                                                              4  ROOTS OUT
                                                                                  ========
```

Fig. 3. A case study of an unstable system of degree 8 with four roots outside the unit circle.

```
Highest Degree=          6                                                     [a]    OK
Degree:                  0     1     2     3     4     5     6                 [b]    OK
===============================================================================
Enter Coefficients:0.083     0  0.25     0 0.333     0     1                   [c]    OK
                         1     0 0.333     0  0.25     0 0.083
---------------------------------------------------------
                     -0.99     0 -0.31     0 -0.22     0     0                        OK
                         0 -0.22     0 -0.31     0 -0.99
---------------------------------------------------
                     0.986     0  0.31     0 0.221     0                              OK
                     0.221     0  0.31     0 0.986
---------------------------------------------
                     0.924     0 0.238     0     0                                    OK
                         0 0.238     0 0.924
----------------------------------------
                     0.853     0 0.219     0                                          OK

===============================================================================  =========
P(i=1..n)            -0.92 -0.91  -0.7 -0.64 -0.41                          -0.278 STABLE
                                                                                 =========
                                                                                 0 ROOTS OUT
                                                                                 =========
```

Fig. 4. A sample worksheet of Jury's method for the numerator of the system of equation (20).

```
Highest Degree=          6                                                     [a]    OK
Degree:                  0     1     2     3     4     5     6                 [b]    OK
===============================================================================
Enter Coefficients: 0.056    0 0.111     0   0.5     0     1                   [c]    OK
                         1    0   0.5     0 0.111     0 0.056
---------------------------------------------------------
                        -1    0 -0.49     0 -0.08     0     0                         OK
                         0 -0.08     0 -0.49     0    -1
---------------------------------------------------
                     0.994    0 0.492     0 0.083     0                               OK
                     0.083    0 0.492     0 0.994
---------------------------------------------
                     0.981    0 0.448     0     0                                     OK
                         0 0.448     0 0.981
----------------------------------------
                     0.962    0  0.44     0                                           OK

===============================================================================  =========
P(i=1..n)            -0.94 -0.94 -0.86 -0.84 -0.44                          -0.322 STABLE
                                                                                 =========
                                                                                 0 ROOTS OUT
                                                                                 =========
```

Fig. 5. A sample worksheet of Jury's method for the denominator of the system of equation (20).

condition is shown in Figs 4 and 5. It can be clearly seen that it is a stable and a minimum-phase system since both tests gave stable results.

5. CONCLUSION

This paper has presented a new approach for the simulation of causal filters that leads to their stability analysis using Jury's method. The power of this tool is the fact that it avoids using a special engineering software package to solve the problem. Instead it uses a well-known and widely used package, Lotus 1-2-3. This worksheet program may be used as a simulation and educational tool. The main features of this tool are its simplicity, flexibility, portability, expendability, and the user-friendly DOS environment under which Lotus 1-2-3 operates. The program is also used to check for minimum-phase condition.

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