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Constant Keplerian Orbit with Non-Central Force Field

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Simple mathematical derivations are presented to show the conditions under which motion with variable angular momentum can result in a constant Keplerian orbit. Derivation of the invariant associated with the differential equation describing the motion is also given in the case of variable angular momentum. The interest of the result is that it shows how to generate periodic solutions of differential equations when friction forces are taken into account.

AUTHOR QUESTIONNAIRE

1. The paper discusses materials/software for a course in:
   Space engineering, dynamics.
2. Students of the following departments are taught in this course:
   Engineering/Science, Mechanical Engineering.
3. Level of the course (year):
   3rd year and 4th year.
4. Mode of presentation:
   Blackboard.
5. Is the material presented in a regular or elective course:
   Regular.
6. Class or hours required to cover the material:
   One-semester course.
7. Student homework or revision hours required for the materials:
   3 to 4 hours per week.
8. Description of the novel aspects present in your paper:
   Periodic elliptical orbits with non-central force field.
9. The standard text recommended in the course, in addition to author's notes:
   J. M. A. Dandy: *Fundamentals of Celestial Mechanics*, 2nd edition, (Willmann-Bell 1988)
10. The material is/is not covered in the text:
    Is not covered in the text.

INTRODUCTION

THE importance of Kepler's laws in celestial mechanics is well known. Kepler's laws, to be discussed later on, are essentially based on a central force field assumption and consequently constant angular momentum. In a recent study [1], two famous problems were proved in a simple manner:

1. That planets pursuing Keplerian elliptical trajectories have accelerations which conform to Newton's central $1/r^2$ equation (central force field);
2. conversely, that planetary orbits must be Keplerian if Newton's central $1/r^2$ equation holds true.

The present study also shows in a simple manner, how a constant Keplerian orbit with variable angular momentum (non-central force field) can result from a motion with friction force and described by a different equation.

$$\ddot{\mathbf{r}} + f\dot{\mathbf{r}} + g\mathbf{r} = 0. \qquad (1)$$

f and g are scalar functions of the variables. Equation (1) was used to describe closed orbits that are conical sections [2–5]. An invariant associated with the one dimensional form of equation (1) was also derived by utilizing a time-dependent canonical transformation [6, 7]. In the present study we discuss the following:

1. The condition of coplanarity of the motion described by equation (1) with variable angular momentum. This second section is added for the sake of completeness and parallels the work given in [4].
2. A simple way to derive the invariant associated with equation (1) in case of variable angular momentum (third section).
3. A simple way to derive the constant Keplerian orbit equation associated with equation (1) in case of variable angular momentum (non-central force field) (fourth section).

Since motion in nature is usually accompanied

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with friction, more attention should be given by experimentalists to include the friction force factor $f \neq 0$ in equation (1) in their studies of periodic orbits.

PLANAR MOTION

As in [4], the vector product of $\mathbf{r}$ with equation (1) is formed. Noting that the angular momentum $\mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}}$, equation (1) yields

$$\dot{\mathbf{L}} + f\mathbf{L} = 0. \tag{2}$$

The vector product of equation (2) with $\mathbf{L}$ gives

$$\dot{\mathbf{L}} \times \mathbf{L} = 0. \tag{3}$$

Equation (3) indicates that for planar motion the vector $\dot{\mathbf{L}} = \dot{l}\hat{\mathbf{L}}$ is in the direction of the unit vector $\hat{\mathbf{L}}$ with

$$f = \frac{-\dot{l}}{l}, \tag{4}$$

$\hat{\mathbf{L}}$ is perpendicular to the plane of motion, l is the magnitude of the angular momentum $\mathbf{L}$.

Equation (1) then takes the form

$$\ddot{\mathbf{r}} - \frac{\dot{l}}{l}\dot{\mathbf{r}} + g\mathbf{r} = 0 \tag{5a}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\mathbf{r}}}{l}\right) = \frac{-g}{l}\mathbf{r}. \tag{5b}$$

It is at this point that we shall depart from the derivation given in [4].

INVARIANT OF PLANAR MOTION

Note that in the following derivation, the amplitude of the angular momentum l is not assumed constant. From the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}(\cos\theta) = -\dot{\theta}\sin\theta \tag{6a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sin\theta) = \dot{\theta}\cos\theta \tag{6b}$$

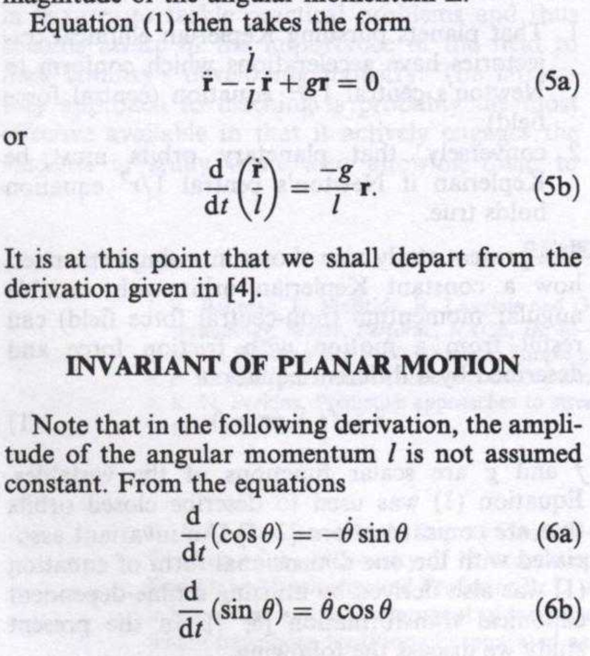

Fig. 1. General arrangement of an ellipse with equation $r = q - \epsilon x$, q = semi-latus rectum, ϵ = eccentricity. The origin of the coordinates is at the right focus, Cartesian coordinates are $x = r\cos\theta$ and $y = r\sin\theta$.

one gets directly by using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ (see Fig. 1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{r}\right) = -\frac{l}{r^3}y = \frac{\dot{x}}{r} - \frac{x}{r^2}\dot{r} \tag{7a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{r}\right) = \frac{l}{r^3}x = \frac{\dot{y}}{r} - \frac{y}{r^2}\dot{r}. \tag{7b}$$

Compare equations (7) with equations (13) and (15) of [1]. Equation (7) give (by noting that $y^2/r^2 = (r^2 - x^2)/r^2$)

$$\left(r\frac{\dot{x}}{l} - x\frac{\dot{r}}{l}\right)^2 + \frac{x^2}{r^2} = 1 \tag{8a}$$

$$\left(r\frac{\dot{y}}{l} - y\frac{\dot{r}}{l}\right)^2 + \frac{y^2}{r^2} = 1. \tag{8b}$$

If l is constant, one gets the invariant derived in [6] and [7].

ORBIT EQUATION

The condition for equation (5b) to represent a constant Keplerian orbit in case of variable angular momentum will now be derived. For this purpose we note that equation (5b) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}}{l}\right) = \frac{-g}{l}x \tag{9a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{y}}{l}\right) = \frac{-g}{l}y \tag{9b}$$

which can be combined to give the differential equation of motion in the radial direction

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{r}}{l}\right) = \frac{-g}{l}r + \frac{\dot{\theta}}{r}. \tag{10}$$

On the other hand, the equation of a Keplerian conic is given by

$$r = \frac{q}{1 + \epsilon\cos\theta} \tag{11}$$

where ϵ = eccentricity of the orbit, q = semi-latus rectum of the conic section, ϵ and q are constant. From equation (11), one can easily derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{r}}{l}\right) = \frac{\epsilon}{q}\dot{\theta}\cos\theta. \tag{12}$$

Comparison of equations (10) and (12) and by making use of equation (11) we get

$$\frac{g}{l}r = \frac{\dot{\theta}}{q} \tag{13}$$

or

$$g = \frac{r\dot{\theta}^2}{q} = \frac{l^2}{r^3 q}. \tag{14}$$

Equations (13) and (14) give the conditions for equation (1) to represent a constant Keplerian orbit described by equation (11), the motion being planar with variable angular momentum.

Equation (14) shows that the force factor g in this case is proportional to the centripetal force $r\dot{\theta}^2$, the proportionality constant being the inverse of the semi-latus rectum of the conic section. Note that by writing

$$f = -\frac{\dot{l}}{l} = -\tfrac{1}{2}(\alpha + 3)\frac{\dot{r}}{r}$$

and $g = \mu r^{\alpha}$, one gets equation (2.1) of [5]. Generally from $l = r^2\dot{\theta}$, one has

$$\frac{\dot{l}}{l} = 2\frac{\dot{r}}{r} + \frac{\ddot{\theta}}{\dot{\theta}}. \quad (15)$$

Equation (8) can be written in the form

$$r\frac{\dot{x}}{l} - x\frac{\dot{r}}{l} = \sin\theta \quad (16a)$$

$$r\frac{\dot{y}}{l} - y\frac{\dot{r}}{l} = \cos\theta \quad (16b)$$

Similarly, equation (11) is equivalent to

$$r = q - \epsilon x \quad (17)$$

$$\frac{\dot{r}}{l} = -\epsilon\frac{\dot{x}}{l} \quad (18)$$

Combining equations (16), (17) and (18) together gives

$$\frac{\dot{r}}{l} = \frac{\epsilon}{q}\sin\theta \quad (19)$$

$$r\frac{\dot{\theta}}{l} = \frac{1}{r} = \frac{1}{q} + \frac{\epsilon}{q}\cos\theta. \quad (20)$$

Equations (19) and (20) give the radial $\dot{r}$ and tangential $r\dot{\theta}$ components of the velocity for an elleptical orbit with variable angular momentum l and should be compared with the results given in [8] and [9] for the case where l is constant.

By introducing the Keplerian force factor K, it is easy to show from equation (14) that we have

$$g = K/r^3 \quad (21)$$

$$K = l^2/q \quad (22)$$

where K is variable when l is variable, equation (22) shows the way the ratio l^2/K behaves in order to obtain a Keplerian orbit with constant semi-latus rectum q, and a constant eccentricity ϵ given by

$$\epsilon = \sqrt{1 + 2(l^2/K)(E/K)}, \quad (23)$$

E is not constant, but the ratio E/l^2 or E/K is constant according to equation (22). E is given by

$$E/l^2 = -(K/l^2)/(2a) = \tfrac{1}{2}(\dot{s}/l)^2 - (K/l^2)/r. \quad (24)$$

E is equivalent to the energy when l is constant, s is the arc length and $\dot{s}$ is the magnitude of the velocity, and we have

$$(\dot{s}/l)^2 = (\dot{r}/l)^2 + (r\dot{\theta}/l)^2 \quad (25a)$$

$$(\dot{s}/l)^2 = (\epsilon/q)^2 + \frac{2\epsilon}{q^2}\cos\theta + \frac{1}{q^2}. \quad (25b)$$

The period for describing an elliptical orbit is given by

$$T = \int_0^{s_e} \frac{ds}{\dot{s}} = \int_0^{2\pi} \frac{d\theta}{\dot{\theta}}, \quad (26)$$

where s_e is the contour of the ellipse, $\dot{\theta} = l/r^2$ and l can be calculated by integration of equation (4) if f is known.

$$l = e^{-\int f\,dt} = \sqrt{r^3 qg}. \quad (27)$$

The area A scanned at time t is given by

$$2A = \int_0^t l\,dt. \quad (28)$$

Kepler's second law appears as a special case of equation (28) when l is constant. From equations (22) and (28) one has

$$\frac{2A}{\sqrt{q}} = \int_0^t \sqrt{K}\,dt. \quad (29)$$

By writing $A_e = \pi a^2\sqrt{1-\epsilon^2}$, $2a =$ major axis of the ellipse $A_e =$ area of the ellipse, and $q = \sqrt{a(1-\epsilon^2)}$, Kepler's third law appears as a special case of equation (29) when K is constant. Another approach to the problem discussed in this study can be found in a recent study given in [10].

DISCUSSION

The conditions under which a periodic motion along an elliptic orbit can take place when friction forces are present ($f \neq 0$ in equation (1)) have been discussed. Kepler's laws appear to correspond to an ideal case for motion with no friction ($f = 0$ in equation (1)). Since motion in nature is usually accompanied with friction, more attention should be given by experimentalists to the study of periodic orbits with $f \neq 0$ in equation (1). Important engineering applications of the results of this study can be found in the design of artificial satellite orbits for instance, as well as in the study of the motion of planets and comets. In problems of dynamics, the study of periodic solutions, or deviation from periodic solutions, for differential equations such as equation (1) is another interesting aspect of this study.

For example, one can take

$$f = \frac{-\dot{l}}{l} = -a$$

with $a > 0$ and constant. Integration gives $l = e^{at}$ and from equation (22) one has

$$K = \frac{e^{2at}}{q}.$$

Consequently, in the presence of friction $f = -a =$

constant, the force factor K increases exponentially with time in order for the orbit to remain elliptical and periodic. Equation (28) shows that the area swept $2A = (e^{at} - 1)/a$ varies also exponentially with time in order for the orbit to remain elliptical.

A more realistic model is to assume a periodic perturbation in l or in K, for instance one can take $l = l_0 + l_1 \cos \omega t$ with $l_1 \ll l_0$. In this case, equation (28) gives for the area swept $2A = \int_0^t (l_0 + l_1 \cos \omega t)\, dt = l_0 t + (l_1/\omega) \sin \omega t$. If T is the period, and if $\omega T = 2n\pi$, where n is an integer, one has

$$\frac{2A_e}{l_0} = T = \text{constant}.$$

Equation (29) shows that in this case $T^2/a^3 =$ constant, which is Kepler's third law.

CONCLUSION

The present study gives a simple way to prove the conditions (equations (4) and (13)) for equation (1) to represent a planar constant Keplerian orbit with variable angular orbit, as well as a simple derivation of the invariant associated with the differential equation describing such a planar motion (equation (8)). Note that we have made no use of Lenz vector or Hamilton vector as in [3] or [4], and the assumptions on the functions f and g as given by equations (4) and (13) appear simple and general. The procedure described in the present study allows one to develop stable periodic solutions for planar differential equations of the form of equation (1) with friction force. Since motion in nature is usually accompanied with friction, practical applications of equation (1) needs further study.

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