Int. J. Engng Ed. Vol. 12, No. 3, pp. 187–198, 1996
Printed in Great Britain.

0949-149X/91 \$3.00+0.00

A New Approach for the Vibration Analysis of Symmetric Mechanical Systems—Part 3: Two-Dimensional Systems*

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In this paper, the final of a three-part series, application of group theory to the free undamped vibration of symmetric mechanical systems is extended to two-dimensional spring–mass idealizations, whose dynamic response students of mechanical engineering (and designers in industry) may sometimes need to evaluate relatively quickly in order to predict the behaviour of real machinery. Examples based on the C_{2v} symmetry of the rectangle, the C_{3v} symmetry of the equilateral triangle, and the C_{4v} symmetry of the square—these three symmetry groups and their properties were described in Part 1—are considered in turn. It is shown that the computational gains of using group theory for such two-dimensional systems (belonging to higher-order symmetry groups) are even greater than in the case of one-dimensional systems that were dealt with in Part 2. As before, the step-by-step presentation of these illustrative examples is primarily geared towards teaching.

INTRODUCTION

IN THE first part [1], idempotents (with the properties of projection operators) for symmetry groups C_{2v}, C_{3v} and C_{4v} were derived from the character tables of these groups. In this paper, these results are applied to four examples of two-dimensional spring–mass models (in which translational motion is confined to, say, the horizontal plane), with the aim of finding the eigenvalues.

The first two examples involve four-mass systems with one degree of freedom per mass, and with configurations belonging to group C_{2v}; in one of these, the (vertical) symmetry planes of the configuration coincide with the position of the masses, while in the other, the two symmetry planes lie between the masses. These two examples are then followed by one of a three-mass system with one degree of freedom per mass, and having a C_{3v} configuration. The last example involves a C_{4v} configuration of four masses, each mass having two degrees of freedom.

Spring deformations are assumed to be very small compared with the distances between the masses, so that the shape of a system configuration does not alter appreciably as a result of any of the deformations to which it may be subjected. Furthermore, the linear form of Hooke's law is assumed to apply throughout.

In all cases, the greatly simplified approach of group theory enables closed-form results for the eigenvalues to be actually obtained through the solution of very simple first- or second-degree equations.

* Paper accepted 15 May 1995.

A C_{2v} SYSTEM: EXAMPLE 1

Consider the C_{2v} configuration depicted in Fig. 1, with the vertical x and y symmetry planes lying between the masses, as shown. The values of the spring constants (k) are as indicated in the figure, while masses m_1, m_2, m_3 and m_4, being all equal to each other, are simply denoted by m. These masses are free to move in only one (guided) direction, as shown in Fig. 1, with x_1, x_2, x_3 and x_4 denoting their respective freedoms.

Symmetry-adapted displacement functions

The basis vectors for the present problem are obtained by applying the idempotents of group C_{2v}—see expressions (3) of [1]—to the functions ϕ_1 ($=x_1$), ϕ_2 ($=x_2$), ϕ_3 ($=x_3$) and ϕ_4 ($=x_4$), as follows:

Subspace U_1

$$\pi_1\phi_1 = \tfrac{1}{4}(E + C_2 + \sigma_x + \sigma_y)\phi_1 = \tfrac{1}{4}(\phi_1 + \phi_3 + \phi_4 + \phi_2) = \pi_1\phi_2 = \pi_1\phi_3 = \pi_1\phi_4$$

$$\Phi = \phi_1 + \phi_2 + \phi_3 + \phi_4 \quad (1)$$

Subspace U_2

$$\pi_2\phi_1 = \tfrac{1}{4}(E + C_2 - \sigma_x - \sigma_y)\phi_1 = \tfrac{1}{4}(\phi_1 + \phi_3 - \phi_4 - \phi_2) = -\pi_2\phi_2 = \pi_2\phi_3 = -\pi_2\phi_4$$

$$\Phi = \phi_1 - \phi_2 + \phi_3 - \phi_4 \quad (2)$$

Subspace U_3

$$\pi_3\phi_1 = \tfrac{1}{4}(E - C_2 + \sigma_x - \sigma_y)\phi_1 = \tfrac{1}{4}(\phi_1 - \phi_3 + \phi_4 - \phi_2) = -\pi_3\phi_2 = -\pi_3\phi_3 = \pi_3\phi_4$$

$$\Phi = \phi_1 - \phi_2 - \phi_3 + \phi_4 \quad (3)$$

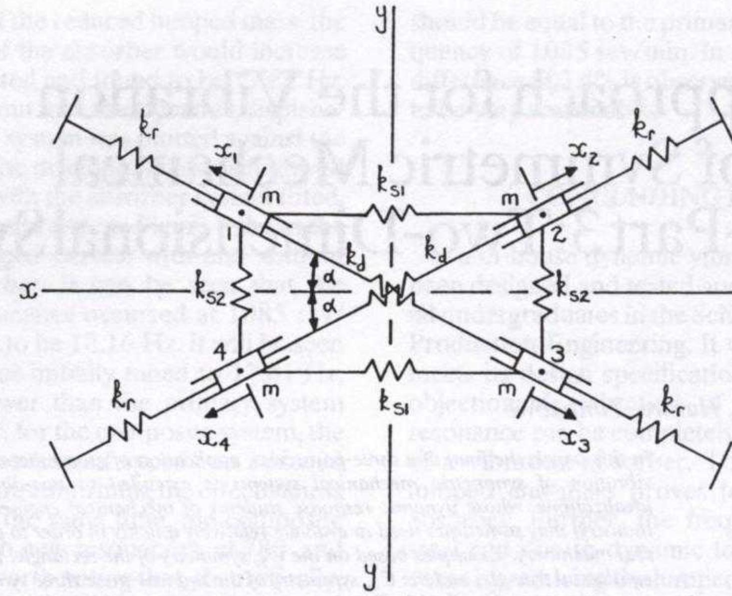

Fig. 1. C_{2v} spring–mass system with $n = 4$. Example 1: symmetry planes lying between the masses.

Subspace U_4

$$\pi_4\phi_1 = \tfrac{1}{4}(E - C_2 - \sigma_x + \sigma_y)\phi_1 = \tfrac{1}{4}(\phi_1 - \phi_3 - \phi_4 + \phi_2) = \pi_4\phi_2 = -\pi_4\phi_3 = -\pi_4\phi_4$$

$$\Phi = \phi_1 + \phi_2 - \phi_3 - \phi_4 \quad (4)$$

Symmetry-adapted stiffness matrices S

These are obtained as explained for the one-dimensional cases [2]. Unit displacements of the masses are applied in accordance with the co-ordinates of the basis vectors (as given by expressions (1)–(4)), as illustrated in Fig. 2. For two-dimensional models in general, the stiffness coefficient s_{ij} ($i = 1, \ldots, r$; $j = 1, \ldots, r$; r is the number of basis vectors spanning the subspace in question) is the component in the direction of any one of the freedoms of Φ_i of the force acting on the mass located at any of the stations of Φ_i, as a result of unit displacements applied in all Φ_j directions at all the stations of Φ_j, while all the freedoms other than those associated with Φ_j, if any, are suppressed. (For a definition of a station of Φ, see [2].) For the present example, where $i = j = 1$ for all the four subspaces, the results are as follows:

Subspace U_1

$$S_1 = [k_r + 2k_d + 2k_{s1}\cos^2\alpha + 2k_{s2}\sin^2\alpha]$$

Subspace U_2

$$S_2 = [k_r + 2k_d]$$

Subspace U_3

$$S_3 = [k_r + 2k_{s2}\sin^2\alpha]$$

Subspace U_4

$$S_4 = [k_r + 2k_{s1}\cos^2\alpha]$$

Eigenvalues

The usual condition [3, 4]

$$|M^{-1}K - \lambda I| = 0 \quad (5)$$

applies, where $\lambda = \omega^2$ (ω being a natural circular frequency of the system), K is the stiffness matrix, M the mass matrix, and I the corresponding identity matrix. As before [2], the subspaces are considered independently of each other, using the associated symmetry-adapted stiffness matrix S in place of the K in the above equation. In general, the mass matrix M to be adopted for a given subspace is the diagonal matrix $[m_{ii}]$ giving the value of the mass at any of the stations of a basis vector Φ_i of the subspace, for $i = 1, \ldots, r$, where r is the dimension of the subspace (i.e. the number of independent basis vectors spanning the subspace). Thus, in the present example, M is one-dimensional and given by

$$M = [m]$$

for all the subspaces. Applying equation (5) to the four subspaces in turn, one obtains the following first-degree characteristic equations in λ, with roots readily following:

Subspace U_1

$$\frac{k_r + 2k_d + 2k_{s1}\cos^2\alpha + 2k_{s2}\sin^2\alpha}{m} - \lambda = 0$$

$$\lambda = \frac{k_r + 2k_d + 2k_{s1}\cos^2\alpha + 2k_{s2}\sin^2\alpha}{m}$$

Subspace U_2

$$\frac{k_r + 2k_d}{m} - \lambda = 0$$

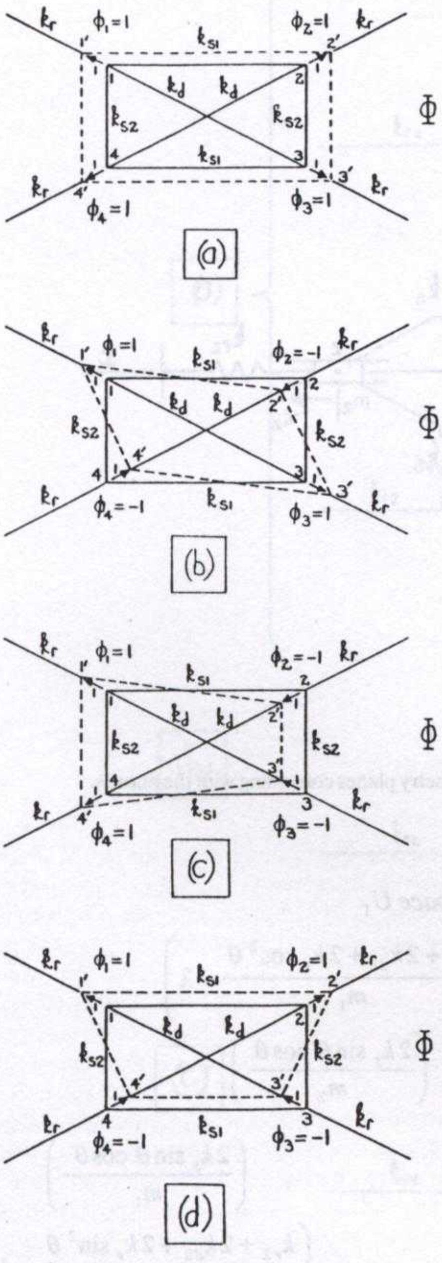

Fig. 2. Unit displacements of masses applied in accordance with the coordinates of the basis vectors for the first C_{2v} example: (a) subspace U_1; (b) subspace U_2; (c) subspace U_3; (d) subspace U_4.

$$\lambda = \frac{k_r + 2k_d}{m}$$

Subspace U_3

$$\frac{k_r + 2k_{s2}\sin^2\alpha}{m} - \lambda = 0$$

$$\lambda = \frac{k_r + 2k_{s2}\sin^2\alpha}{m}$$

Subspace U_4

$$\frac{k_r + 2k_{s1}\cos^2\alpha}{m} - \lambda = 0$$

$$\lambda = \frac{k_r + 2k_{s1}\cos^2\alpha}{m}$$

The above four values of λ are the eigenvalues sought in the original problem. Thus, by means of group theory, the problem has been reduced to solving a set of four very simple first-degree equations which are independent of each other, instead of a fourth-degree polynomial equation in λ that is yielded by conventional considerations.

A C_{2v} SYSTEM: EXAMPLE 2

For the C_{2v} configuration of Fig. 3, the vertical x and y symmetry planes coincide with the positions of the masses. These masses have one degree of freedom each, in the guided directions denoted by x_1, x_2, x_3 and x_4, as shown in the figure.

Symmetry-adapted displacement functions

Applying the C_{2v} idempotents [1] to the variables $\phi_1(=x_1), \phi_2(=x_2), \phi_3(=x_3)$ and $\phi_4(=x_4)$, the following symmetry-adapted displacement functions—basis vectors for the respective subspaces—are obtained:

Subspace U_1

$$\pi_1\phi_1 = \tfrac{1}{4}(E + C_2 + \sigma_x + \sigma_y)\phi_1 = \tfrac{1}{4}(\phi_1 + \phi_3 + \phi_3 + \phi_1) = \tfrac{1}{2}(\phi_1 + \phi_3) = \pi_1\phi_3$$

$$\pi_1\phi_2 = \tfrac{1}{4}(E + C_2 + \sigma_x + \sigma_y)\phi_2 = \tfrac{1}{4}(\phi_2 + \phi_4 + \phi_2 + \phi_4) = \tfrac{1}{2}(\phi_2 + \phi_4) = \pi_1\phi_4$$

$$\Phi_1 = \phi_1 + \phi_3 \quad (6a)$$

$$\Phi_2 = \phi_2 + \phi_4 \quad (6b)$$

Subspace U_2

$$\pi_2\phi_1 = \tfrac{1}{4}(E + C_2 - \sigma_x - \sigma_y)\phi_1 = \tfrac{1}{4}(\phi_1 + \phi_3 - \phi_3 - \phi_1) = 0 = \pi_2\phi_3$$

$$\pi_2\phi_2 = \tfrac{1}{4}(E + C_2 - \sigma_x - \sigma_y)\phi_2 = \tfrac{1}{4}(\phi_2 + \phi_4 - \phi_2 - \phi_4) = 0 = \pi_2\phi_4$$

U_2 is a null subspace.

Subspace U_3

$$\pi_3\phi_1 = \tfrac{1}{4}(E - C_2 + \sigma_x - \sigma_y)\phi_1 = \tfrac{1}{4}(\phi_1 - \phi_3 + \phi_3 - \phi_1) = 0 = \pi_3\phi_3$$

$$\pi_3\phi_2 = \tfrac{1}{4}(E - C_2 + \sigma_x - \sigma_y)\phi_2 = \tfrac{1}{4}(\phi_2 - \phi_4 + \phi_2 - \phi_4) = \tfrac{1}{2}(\phi_2 - \phi_4) = -\pi_3\phi_4$$

$$\Phi = \phi_2 - \phi_4 \quad (7)$$

Subspace U_4

$$\pi_4\phi_1 = \tfrac{1}{4}(E - C_2 - \sigma_x + \sigma_y)\phi_1 = \tfrac{1}{4}(\phi_1 - \phi_3 - \phi_3 + \phi_1) = \tfrac{1}{2}(\phi_1 - \phi_3) = -\pi_4\phi_3$$

$$\pi_4\phi_2 = \tfrac{1}{4}(E - C_2 - \sigma_x + \sigma_y)\phi_2 = \tfrac{1}{4}(\phi_2 - \phi_4 - \phi_2 + \phi_4) = 0 = \pi_4\phi_4$$

$$\Phi = \phi_1 - \phi_3 \quad (8)$$

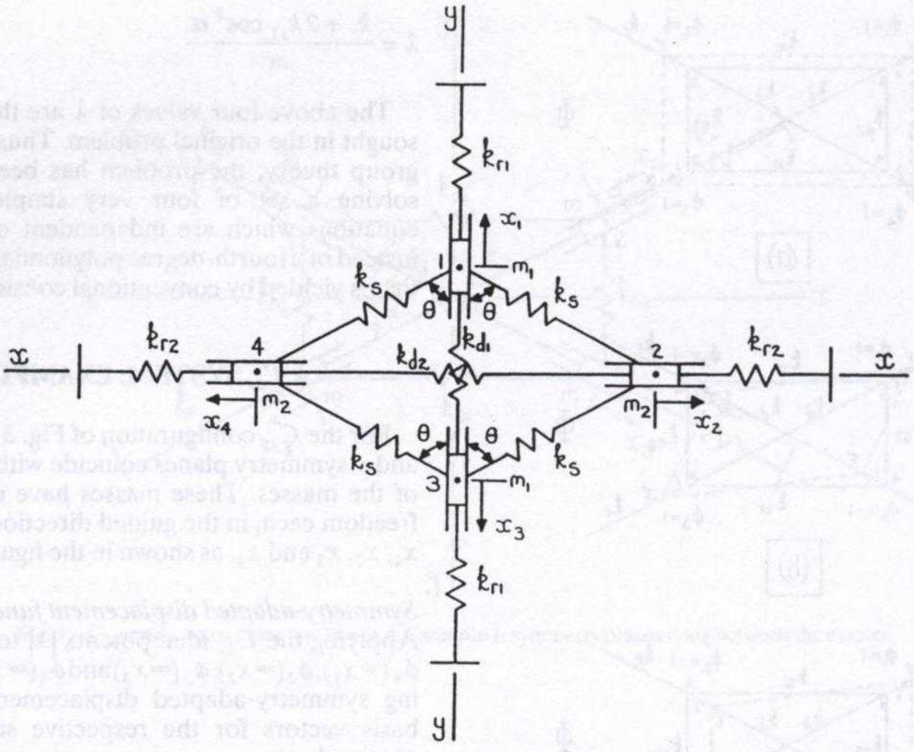

Fig. 3. C_{2v} spring–mass system with $n = 4$. Example 2: symmetry planes coinciding with the masses.

Symmetry-adapted stiffness matrices

With reference to Fig. 4, the coefficients of the stiffness matrices for the various subspaces are obtained as explained above. The resulting stiffness matrices are as follows:

Subspace U_1

$$S_1 = \begin{bmatrix} (k_{r1} + 2k_{d1} + 2k_s \cos^2\theta) & (2k_s \sin\theta\cos\theta) \\ (2k_s \sin\theta\cos\theta) & (k_{r2} + 2k_{d2} + 2k_s \sin^2\theta) \end{bmatrix}$$

Subspace U_3

$$S_3 = [k_{r2} + 2k_s \sin^2\theta]$$

Subspace U_4

$$S_4 = [k_{r1} + 2k_s \cos^2\theta]$$

Eigenvalues

Using expression (5), with the appropriate stiffness matrix for the subspace in question in place of K, and the mass matrices M_1 (for subspace U_1), M_3 (for subspace U_3) and M_4 (for subspace U_4), where

$$M_1 = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}; \quad M_3 = [m_2]; \quad \text{and} \quad M_4 = [m_1]$$

in accordance with the definition for M given earlier, the following characteristic equations and their roots ensue:

Subspace U_1

$$\begin{vmatrix} \left(\dfrac{k_{r1} + 2k_{d1} + 2k_s\cos^2\theta}{m_1} - \lambda\right) & \left(\dfrac{2k_s \sin\theta\cos\theta}{m_1}\right) \\ \left(\dfrac{2k_s \sin\theta\cos\theta}{m_2}\right) & \left(\dfrac{k_{r2} + 2k_{d2} + 2k_s\sin^2\theta}{m_2} - \lambda\right) \end{vmatrix} = 0$$

$$\lambda = \tfrac{1}{2}\left(-b \pm \sqrt{b^2 - 4c}\right)$$

where

$$b = -\frac{1}{m_1 m_2}\{(k_{r1} + 2k_{d1} + 2k_s\cos^2\theta)m_2 + (k_{r2} + 2k_{d2} + 2k_s\sin^2\theta)m_1\}$$

$$c = \frac{1}{m_1 m_2}\{(k_{r1} + 2k_{d1})(k_{r2} + 2k_{d2}) + 2k_s \times [(k_{r1} + 2k_{d1})\sin^2\theta + (k_{r2} + 2k_{d2})\cos^2\theta]\}$$

Subspace U_3

$$\frac{k_{r2} + 2k_s \sin^2\theta}{m_2} - \lambda = 0$$

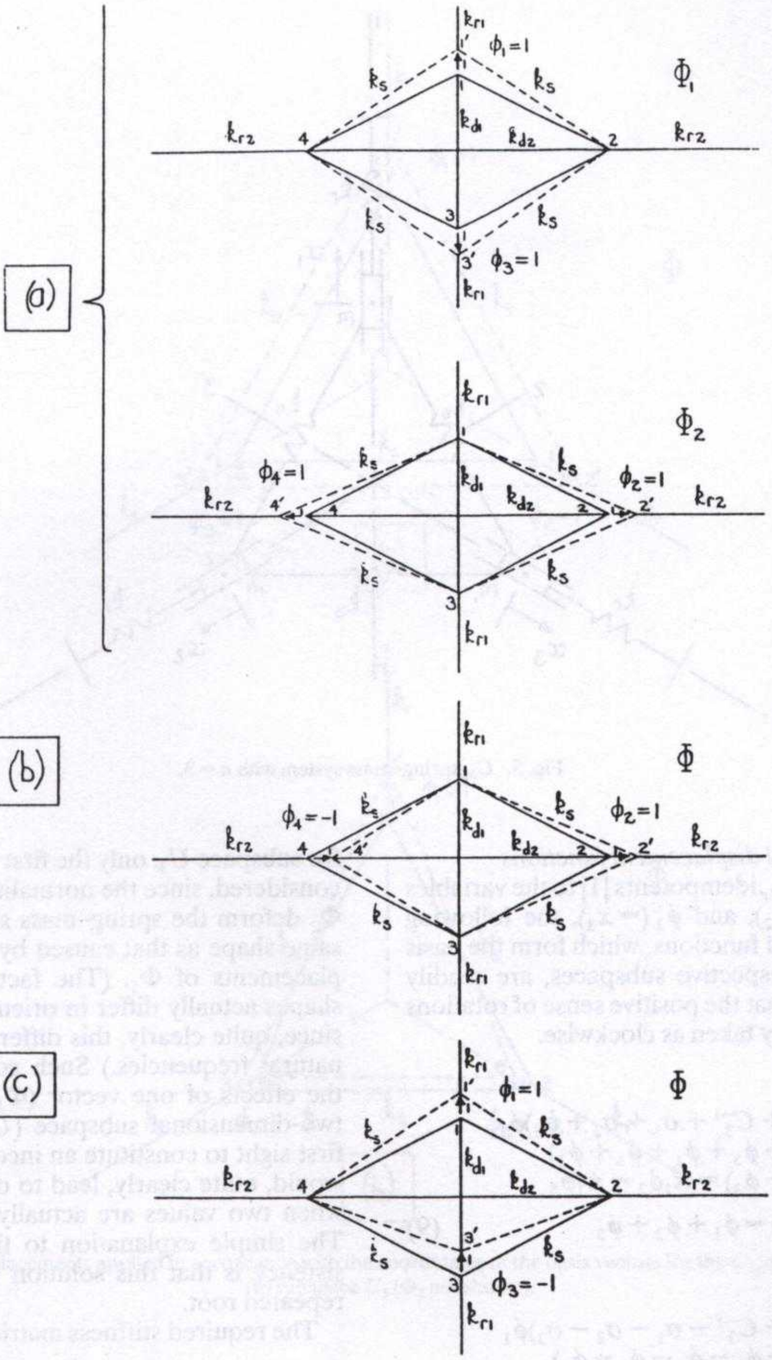

Fig. 4. Unit displacements of masses applied in accordance with the coordinates of the basis vectors for the second C_{2v} example: (a) subspace U_1; (b) subspace U_3; (c) subspace U_4.

$$\lambda = \frac{k_{r2} + 2k_s \sin^2 \theta}{m_2}$$

Subspace U_4

$$\frac{k_{r1} + 2k_s \cos^2 \theta}{m_1} - \lambda = 0$$

$$\lambda = \frac{k_{r1} + 2k_s \cos^2 \theta}{m_1}$$

A C_{3v} SYSTEM

Figure 5 shows a C_{3v} configuration, with the equilibrium positions 1, 2 and 3 of the oscillating masses lying at the vertices of an equilateral triangle. The three masses are all equal to each other (i.e. $m_1 = m_2 = m_3 = m$); constrained to move only along the line passing through the centroid of the equilateral triangle and its equilibrium position, each mass m_i thus has one degree of freedom x_i ($i = 1, 2, 3$), as shown in Fig. 5.

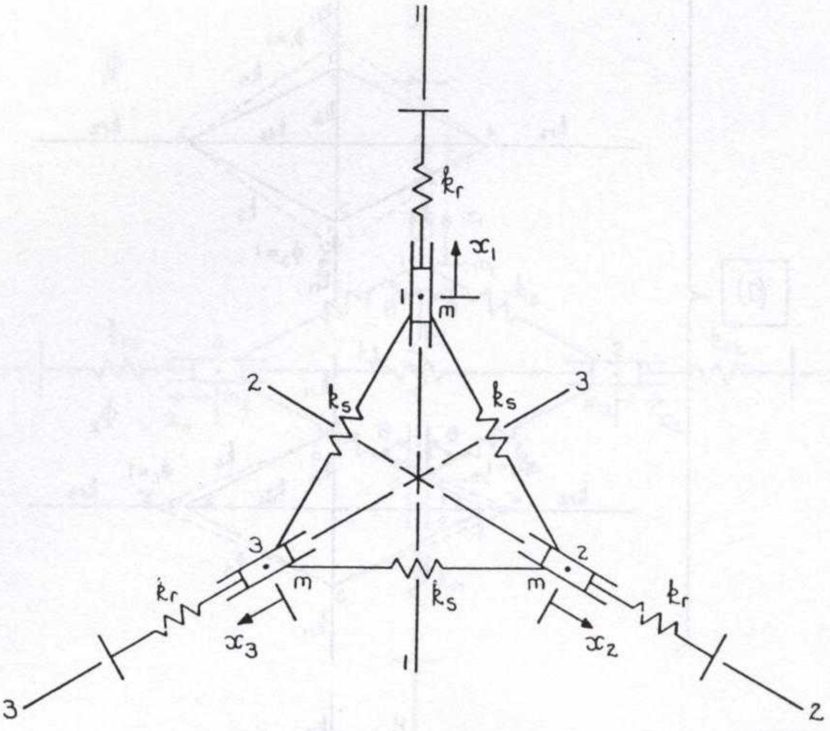

Fig. 5. C_{3v} spring–mass system with $n = 3$.

Symmetry-adapted displacement functions

Applying the C_{3v} idempotents [1] to the variables ϕ_1 ($= x_1$), ϕ_2 ($= x_2$), and ϕ_3 ($= x_3$), the following symmetry-adapted functions, which form the basis vectors for the respective subspaces, are readily obtained, noting that the positive sense of rotations has been arbitrarily taken as clockwise.

Subspace U_1

$$\begin{aligned}\pi_1\phi_1 &= \tfrac{1}{6}(E + C_3 + C_3^{-1} + \sigma_1 + \sigma_2 + \sigma_3)\phi_1\\ &= \tfrac{1}{6}(\phi_1 + \phi_2 + \phi_3 + \phi_1 + \phi_3 + \phi_2)\\ &= \tfrac{1}{3}(\phi_1 + \phi_2 + \phi_3) = \pi_1\phi_2 = \pi_1\phi_3\end{aligned}$$

$$\Phi = \phi_1 + \phi_2 + \phi_3 \qquad (9)$$

Subspace U_2

$$\begin{aligned}\pi_2\phi_1 &= \tfrac{1}{6}(E + C_3 + C_3^{-1} - \sigma_1 - \sigma_2 - \sigma_3)\phi_1\\ &= \tfrac{1}{6}(\phi_1 + \phi_2 + \phi_3 - \phi_1 - \phi_3 - \phi_2)\\ &= 0 = \pi_2\phi_2 = \pi_2\phi_3\end{aligned}$$

U_2 is a null subspace.

Subspace U_3

$$\begin{aligned}\pi_3\phi_1 &= \tfrac{1}{3}(2E - C_3 - C_3^{-1})\phi_1 = \tfrac{1}{3}(2\phi_1 - \phi_2 - \phi_3)\\ \pi_3\phi_2 &= \tfrac{1}{3}(2E - C_3 - C_3^{-1})\phi_2 = \tfrac{1}{3}(2\phi_2 - \phi_3 - \phi_1)\\ \pi_3\phi_3 &= \tfrac{1}{3}(2E - C_3 - C_3^{-1})\phi_3 = \tfrac{1}{3}(2\phi_3 - \phi_1 - \phi_2)\\ &= -\pi_3(\phi_1 + \phi_2)\end{aligned}$$

$$\Phi_1 = \phi_1 - \tfrac{1}{2}\phi_2 - \tfrac{1}{2}\phi_3 \qquad (10a)$$

$$\Phi_2 = \phi_2 - \tfrac{1}{2}\phi_3 - \tfrac{1}{2}\phi_1 \qquad (10b)$$

Symmetry-adapted stiffness matrices

The stiffness coefficients for these are obtained as explained previously. Normalized displacements are applied as illustrated in Fig. 6, noting that for subspace U_3, only the first vector (Φ_1) has been considered, since the normalized displacements of Φ_2 deform the spring–mass system to exactly the same shape as that caused by the normalized displacements of Φ_1. (The fact that the deformed shapes actually differ in orientation is immaterial, since, quite clearly, this difference does not affect natural frequencies.) Such consideration of only the effects of one vector (Φ_1), for an essentially two-dimensional subspace (U_3), might appear at first sight to constitute an inconsistency, since this would, quite clearly, lead to only one value for λ, when two values are actually expected from U_3. The simple explanation to this apparent inconsistency is that this solution would, in fact, be a repeated root.

The required stiffness matrices are as follows:

Subspace U_1

$$S_1 = [k_r + 4k_s \cos^2 30°]$$

Subspace U_3

$$S_3 = [k_r + k_s \cos^2 30°] \qquad \text{twice}$$

Eigenvalues

From expression (5), noting that

$$M_1 = M_3 = [m],$$

the following characteristic equations, and their roots, are readily obtained:

Subspace U_1

$$\frac{k_r + 4k_s \cos^2 30°}{m} - \lambda = 0$$

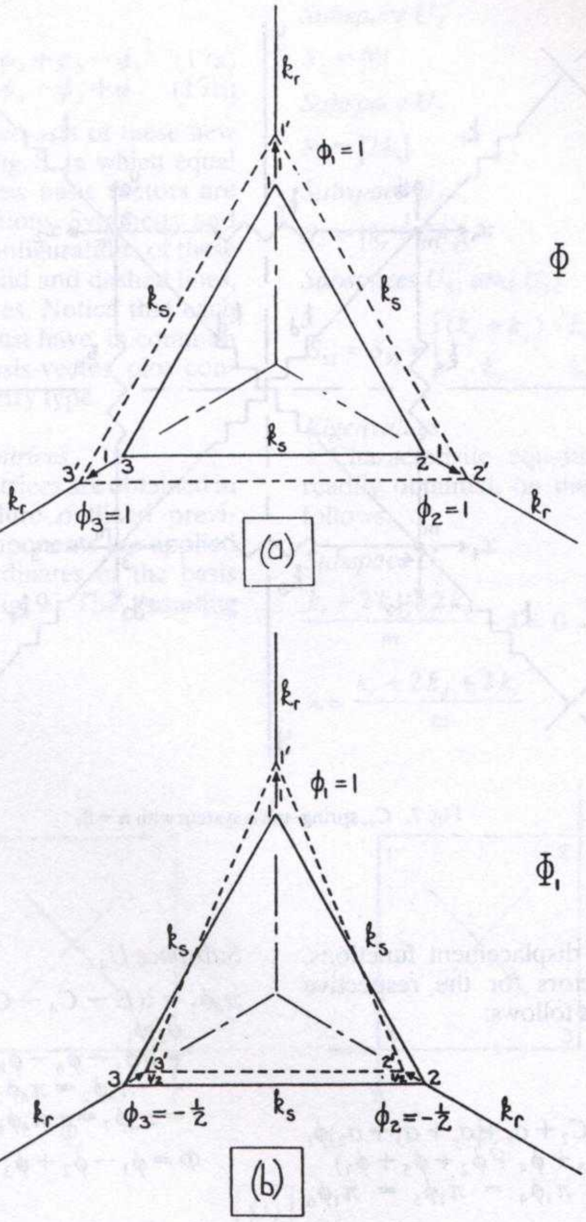

Fig. 6. Normalized displacements applied in accordance with the coordinates of the basis vectors for the C_{3v} example: (a) subspace U_1; (b) subspace U_3 (Φ_2 not shown).

$$\lambda = \frac{k_r + 4k_s \cos^2 30^\circ}{m}$$

Subspace U_3

$$\frac{k_r + k_s \cos^2 30^\circ}{m} - \lambda = 0 \qquad \text{twice}$$

$$\lambda = \frac{k_r + k_s \cos^2 30^\circ}{m} \qquad \text{twice}$$

A C_{4v} SYSTEM

The final example involves four equal masses occupying the corners of a square, and interconnected through springs, the whole arrangement having a C_{4v} configuration [1], as shown in Fig. 7. Each mass is free to move in any direction in the plane of the configuration, so that there are eight independent freedoms in total. The system of freedoms shown in Fig. 7, and the assumed directions of their positive displacements, have been chosen to conform to the C_{4v} symmetry of the physical configuration.

Symmetry-adapted displacement functions

The C_{4v} idempotents—expressions (5) of [1]—are applied to the variables $\phi_1 (= x_1)$, $\phi_2 (= x_2)$, $\phi_3 (= x_3)$, $\phi_4 (= x_4)$, $\phi_5 (= y_1)$, $\phi_6 (= y_2)$, $\phi_7 (= y_3)$ and $\phi_8 (= y_4)$. As before, the positive directions of rotation operations are arbitrarily taken as clock-

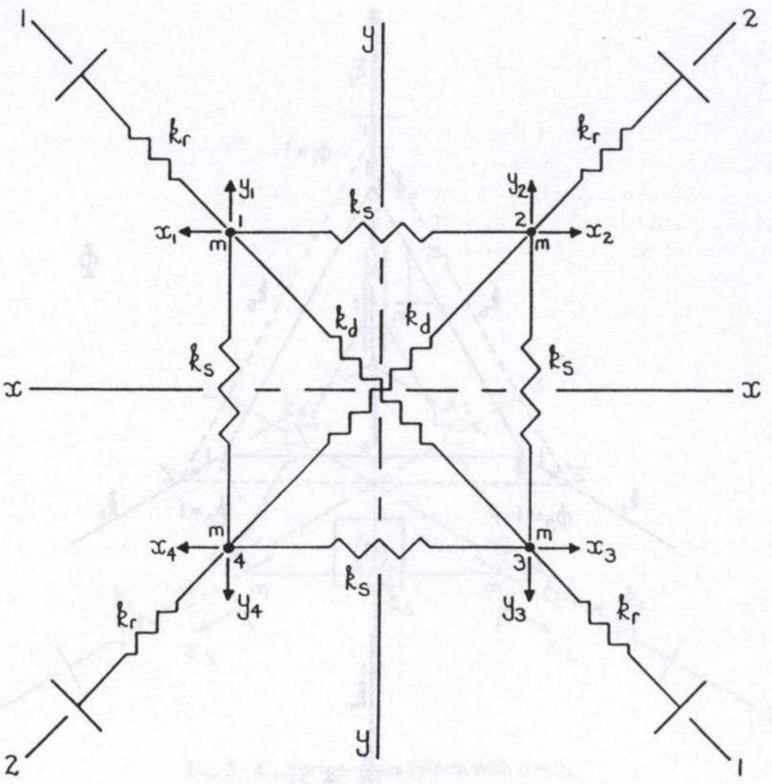

Fig. 7. C_{4v} spring–mass system with $n = 8$.

wise. Symmetry-adapted displacement functions, which are the basis vectors for the respective subspaces, are obtained as follows:

Subspace U_1

$$\begin{aligned}\pi_1\phi_1 &= \tfrac{1}{8}(E + C_4 + C_4^{-1} + C_2 + \sigma_x + \sigma_y + \sigma_1 + \sigma_2)\phi_1 \\ &= \tfrac{1}{8}(\phi_1 + \phi_6 + \phi_8 + \phi_3 + \phi_4 + \phi_2 + \phi_5 + \phi_7) \\ &= \pi_1\phi_2 = \pi_1\phi_3 = \pi_1\phi_4 = \pi_1\phi_5 = \pi_1\phi_6 \\ &= \pi_1\phi_7 = \pi_1\phi_8\end{aligned}$$

$$\Phi = \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6 + \phi_7 + \phi_8 \tag{11}$$

Subspace U_2

$$\begin{aligned}\pi_2\phi_1 &= \tfrac{1}{8}(E + C_4 + C_4^{-1} + C_2 - \sigma_x - \sigma_y - \sigma_1 - \sigma_2)\phi_1 \\ &= \tfrac{1}{8}(\phi_1 + \phi_6 + \phi_8 + \phi_3 - \phi_4 - \phi_2 - \phi_5 - \phi_7) \\ &= -\pi_2\phi_2 = \pi_2\phi_3 = -\pi_2\phi_4 = -\pi_2\phi_5 = \pi_2\phi_6 \\ &= -\pi_2\phi_7 = \pi_2\phi_8\end{aligned}$$

$$\Phi = \phi_1 - \phi_2 + \phi_3 - \phi_4 - \phi_5 + \phi_6 - \phi_7 + \phi_8 \tag{12}$$

Subspace U_3

$$\begin{aligned}\pi_3\phi_1 &= \tfrac{1}{8}(E - C_4 - C_4^{-1} + C_2 + \sigma_x + \sigma_y - \sigma_1 - \sigma_2)\phi_1 \\ &= \tfrac{1}{8}(\phi_1 - \phi_6 - \phi_8 + \phi_3 + \phi_4 + \phi_2 - \phi_5 - \phi_7) \\ &= \pi_3\phi_2 = \pi_3\phi_3 = \pi_3\phi_4 = -\pi_3\phi_5 = -\pi_3\phi_6 \\ &= -\pi_3\phi_7 = -\pi_3\phi_8\end{aligned}$$

$$\Phi = \phi_1 + \phi_2 + \phi_3 + \phi_4 - \phi_5 - \phi_6 - \phi_7 - \phi_8 \tag{13}$$

Subspace U_4

$$\begin{aligned}\pi_4\phi_1 &= \tfrac{1}{8}(E - C_4 - C_4^{-1} + C_2 - \sigma_x - \sigma_y + \sigma_1 + \sigma_2)\phi_1 \\ &= \tfrac{1}{8}(\phi_1 - \phi_6 - \phi_8 + \phi_3 - \phi_4 - \phi_2 + \phi_5 + \phi_7) \\ &= -\pi_4\phi_2 = \pi_4\phi_3 = -\pi_4\phi_4 = \pi_4\phi_5 = -\pi_4\phi_6 \\ &= \pi_4\phi_7 = -\pi_4\phi_8\end{aligned}$$

$$\Phi = \phi_1 - \phi_2 + \phi_3 - \phi_4 + \phi_5 - \phi_6 + \phi_7 - \phi_8 \tag{14}$$

Subspace U_5

$$\begin{aligned}\pi_5\phi_1 &= \tfrac{1}{2}(E - C_2)\phi_1 = \tfrac{1}{2}(\phi_1 - \phi_3) = -\pi_5\phi_3 \\ \pi_5\phi_2 &= \tfrac{1}{2}(E - C_2)\phi_2 = \tfrac{1}{2}(\phi_2 - \phi_4) = -\pi_5\phi_4 \\ \pi_5\phi_5 &= \tfrac{1}{2}(E - C_2)\phi_5 = \tfrac{1}{2}(\phi_5 - \phi_7) = -\pi_5\phi_7 \\ \pi_5\phi_6 &= \tfrac{1}{2}(E - C_2)\phi_6 = \tfrac{1}{2}(\phi_6 - \phi_8) = -\pi_5\phi_8\end{aligned}$$

$$\Phi_1 = \phi_1 - \phi_3 \tag{15a}$$
$$\Phi_2 = \phi_2 - \phi_4 \tag{15b}$$
$$\Phi_3 = \phi_5 - \phi_7 \tag{15c}$$
$$\Phi_4 = \phi_6 - \phi_8 \tag{15d}$$

Subspace U_5 can be further decomposed into two independent two-dimensional subspaces U_{51} and U_{52}, spanned by new basis vectors obtained by linearly combining the above vectors in such a way as to form two orthogonal sets, as follows:

Subspace U_{51}

$$\Phi'_1 = \Phi_1 + \Phi_3 = \phi_1 - \phi_3 + \phi_5 - \phi_7 \tag{16a}$$

$$\Phi'_2 = \Phi_2 - \Phi_4 = \phi_2 - \phi_4 - \phi_6 + \phi_8 \tag{16b}$$

Subspace U_{52}

$$\Phi_1' = \Phi_2 + \Phi_4 = \phi_2 - \phi_4 + \phi_6 - \phi_8 \quad (17a)$$
$$\Phi_2' = \Phi_1 - \Phi_3 = \phi_1 - \phi_3 - \phi_5 + \phi_7 \quad (17b)$$

The orthogonality of the two sets of these new basis vectors is depicted in Fig. 8, in which equal coordinates of each of the new basis vectors are plotted at their respective stations. Symmetry and antisymmetry planes of the configurations of these vector plots are marked as solid and dashed lines, respectively, inside the squares. Notice that each new subspace (U_{51} or U_{52}) must have, in common with any other subspace, basis-vector plot configurations of the same symmetry type.

Symmetry-adapted stiffness matrices

The coefficients of these matrices are obtained in accordance with the procedure outlined previously. Unit displacement components are applied in accordance with the coordinates of the basis vectors, as illustrated in Fig. 9. The resulting matrices are as follows:

Subspace U_1

$$S_1 = [k_r + 2k_d + 2k_s]$$

Subspace U_2

$$S_2 = [0]$$

Subspace U_3

$$S_3 = [2k_s]$$

Subspace U_4

$$S_4 = [k_r + 2k_d]$$

Subspaces U_{51} and U_{52}

$$S_{51} = S_{52} = \begin{bmatrix} (k_r + k_s) & k_s \\ k_s & k_s \end{bmatrix}$$

Eigenvalues

Characteristic equations and their roots are readily obtained, on the basis of relation (5), as follows:

Subspace U_1

$$\frac{k_r + 2k_d + 2k_s}{m} - \lambda = 0$$

$$\lambda = \frac{k_r + 2k_d + 2k_s}{m}$$

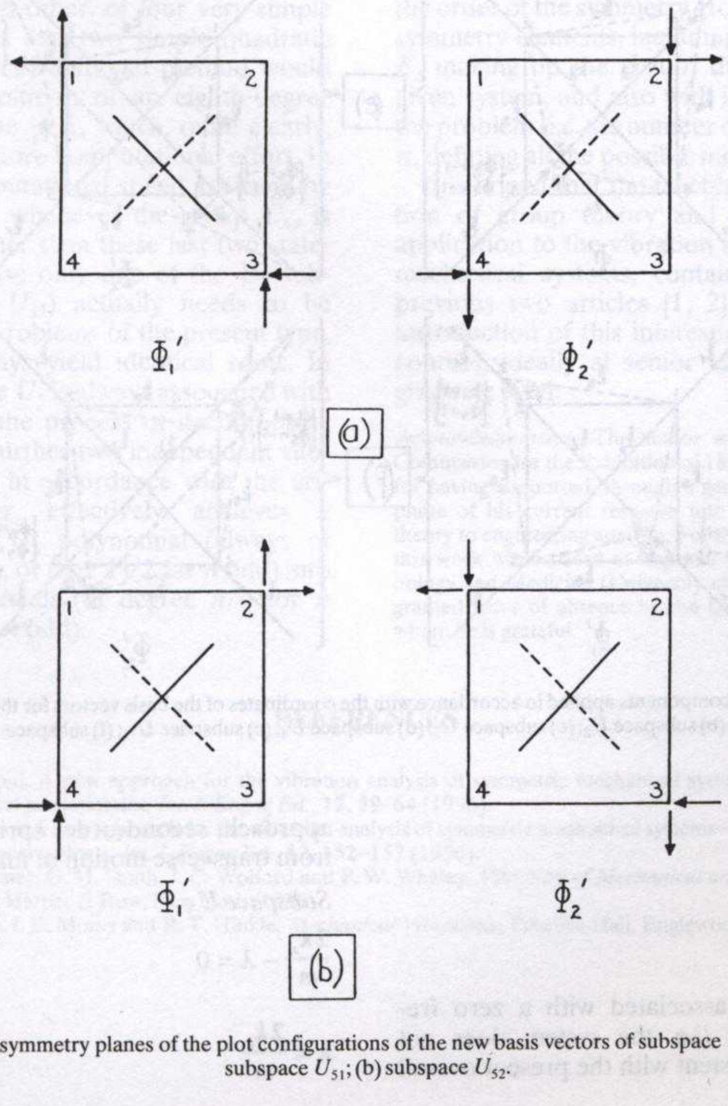

Fig. 8. Symmetry and antisymmetry planes of the plot configurations of the new basis vectors of subspace U_5, for the C_{4v} example: (a) subspace U_{51}; (b) subspace U_{52}.

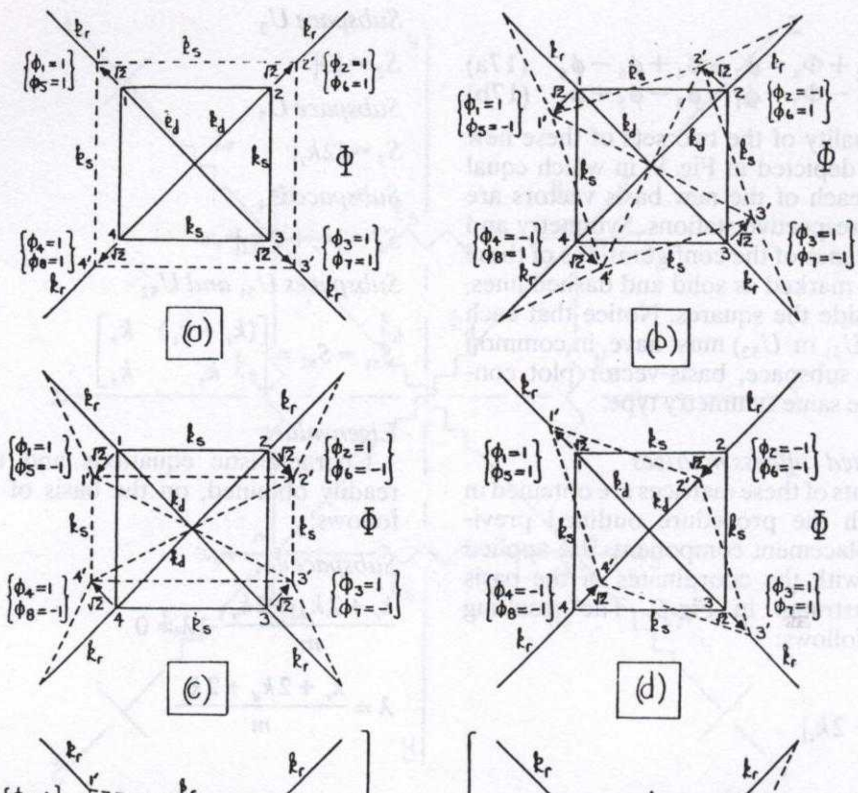

Fig. 9. Unit displacement components applied in accordance with the coordinates of the basis vectors for the C_{4v} example: (a) subspace U_1; (b) subspace U_2; (c) subspace U_3; (d) subspace U_4; (e) subspace U_{51}; (f) subspace U_{52}.

Subspace U_2

$$\frac{0}{m} - \lambda = 0$$

$$\lambda = 0$$

This subspace is associated with a zero frequency of vibration (i.e. the system does not vibrate at all if, consistent with the present overall approach, second-order spring extensions arising from transverse motion of masses are neglected).

Subspace U_3

$$\frac{2k_s}{m} - \lambda = 0$$

$$\lambda = \frac{2k_s}{m}$$

Subspace U_4

$$\frac{k_r + 2k_d}{m} - \lambda = 0$$

$$\lambda = \frac{k_r + 2k_d}{m}$$

Subspace U_{51} *or* U_{52}

$$\begin{vmatrix} \left(\frac{k_r + k_s}{m} - \lambda\right) & \frac{k_s}{m} \\ \frac{k_s}{m} & \left(\frac{k_s}{m} - \lambda\right) \end{vmatrix} = 0$$

$$\lambda = \tfrac{1}{2}\left(-b \pm \sqrt{b^2 - 4c}\right)$$

where

$$b = -\frac{k_r + 2k_s}{m} \quad \text{and} \quad c = \frac{k_r k_s}{m^2}$$

Note that the original problem (with $n = 8$ degrees of freedom) has been decomposed into six independent subspaces, requiring the solution, independently of each other, of four very simple first-degree equations and two simple quadratic equations in λ. The conventional method would have required the solution of an eighth-degree characteristic equation in λ, which, quite clearly, implies many times more computational effort. In fact, the gain in computational speed achieved by use of group theory, whenever the group C_{4v} is involved, is even higher than these last two statements suggest, because only one of the U_5 subspaces (i.e. U_{51} or U_{52}) actually needs to be considered, since in problems of the present type, these subspaces always yield identical roots. In other words, subspace U_5 is always associated with repeated roots, and the process of decomposing this subspace into a further two independent subspaces U_{51} and U_{52}, in accordance with the criterion given earlier, effectively achieves a factorization of the U_5 polynomial (always of degree $n/2$ for n even, or $(n-1)/2$ for n odd) into two identical polynomials (of degree $n/4$ for n even, or $(n-1)/4$ for n odd).

CONCLUSIONS

In this paper, group theory has been applied to the determination of eigenvalues for the vibration of two-dimensional spring–mass models. The procedure has been illustrated through the consideration of two examples with differently orientated C_{2v} configurations, a third example with C_{3v} symmetry, and a final example featuring C_{4v} symmetry, the symmetry groups C_{2v}, C_{3v} and C_{4v} and their properties having been described in an earlier article [1].

While considerable reductions in computational effort were also noted for one-dimensional mechanical systems of the C_2 symmetry group [2], the computational gains of the group-theoretic approach (over the conventional method for eigenvalue determination) are even larger in the case of two-dimensional systems belonging to the higher-order symmetry groups C_{2v}, C_{3v} and C_{4v}. Of all the six examples that have been considered between this and the previous article [2], the particularly efficient decomposition of the group C_{4v}, at least for $n \geqslant 8$, is noteworthy.

In general, the computational merit of the group-theoretic method becomes greater with increase in the order of the symmetry group (i.e. the number of symmetry elements, including the identity element E, making up the group) that is applicable to a given system, and also with increase in the size of the problem (i.e. the number of degrees of freedom, n, defining all the possible motion of a system).

It is hoped that the teaching-oriented presentation of group theory and its highly beneficial application to the vibration analysis of symmetric mechanical systems, contained in this and the previous two articles [1, 2], will encourage the introduction of this interesting topic into existing courses, ideally at senior undergradute or post-graduate level.

Acknowledgements—The author wishes to thank the Royal Commission for the Exhibition of 1851 of the United Kingdom, for having supported, through a generous fellowship, the first phase of his current research into the application of group theory to engineering analysis. For the purposes of carrying out this work while based at Imperial College of Science, Technology and Medicine (University of London), the author was granted leave of absence by the University of Zimbabwe, to whom he is grateful.

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