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A New Approach for the Vibration Analysis of Symmetric Mechanical Systems—Part 2: One-Dimensional Systems*

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Basic concepts of the theory of symmetry groups and their representations were outlined in Part 1. In the present paper, the theoretical results for the simple group C_2 are applied to the determination of eigenvalues for two examples of freely vibrating, undamped mechanical systems with a single plane of symmetry, and extending in one spatial dimension. These comprise the torsional motion of a shaft–disc system, and the extensional oscillations of a spring–mass assembly, two models which are quite familiar to students taking a course in mechanical engineering vibrations. The systematic simplification of the solution of these problems by the group-theoretic approach is clearly illustrated.

INTRODUCTION

TWO examples of one-dimensional symmetric mechanical systems are considered in turn. These comprise the rotational vibrations of a shaft-disc system, and the extensional vibrations of a spring–mass system, the first having an even number of degrees of freedom ($n = 6$), and the second an odd number ($n = 5$). Both configurations belong to the symmetry group C_2, which was described in the first article [1].

Using the idempotents for group C_2 which were derived in the first paper, basis vectors (i.e. symmetry-adapted functions) spanning the respective subspaces of each of the two problems are deduced. These enable symmetry-adapted stiffness matrices (corresponding to the symmetric and antisymmetric subspaces) to be derived, leading to two independent characteristic equations. If n is even, these equations are both of degree $n/2$; if n is odd, one equation will be of degree $(n + 1)/2$, while the other will be of degree $(n - 1)/2$. In either case, the computational effort that would be incurred in solving for the eigenvalues is considerably less than if a polynomial equation of full degre n (as yielded by the conventional method) were to be solved.

A SHAFT–DISC TORSIONAL SYSTEM

Consider a shaft with both ends fixed, and having six discs attached to it in a symmetrical configuration, as shown in Fig. 1. The system has six degrees of freedom, describing the rotational motions θ_1, θ_2, θ_3, θ_4, θ_5 and θ_6 of discs 1, 2, 3, 4, 5 and 6. The mass moments of inertia (I) of the discs, and the torsional constants (k) of the various shaft intervals are as shown in Fig. 1.

Symmetry-adapted rotation functions

The basis vectors (i.e. symmetry-adapted rotation functions) of the two subspaces U_1 and U_2 associated with the group C_2 are obtained by applying the respective idempotents (π_1 in the case of U_1, and π_2 in the case of U_2) to the system functions $\phi_1(=\theta_1)$, $\phi_2(=\theta_2)$, $\phi_3(=\theta_3)$, $\phi_4(=\theta_4)$, $\phi_5(=\theta_5)$ and $\phi_6(=\theta_6)$, as shown below. The idempotents π_1 and π_2 for group C_2 have been defined in equation (2) of the first article [1].

Subspace U_1

$$\pi_1\phi_1 = \tfrac{1}{2}(E + C_2)\phi_1 = \tfrac{1}{2}(\phi_1 + \phi_6) = \pi_1\phi_6$$
$$\pi_1\phi_2 = \tfrac{1}{2}(E + C_2)\phi_2 = \tfrac{1}{2}(\phi_2 + \phi_5) = \pi_1\phi_5$$
$$\pi_1\phi_3 = \tfrac{1}{2}(E + C_2)\phi_3 = \tfrac{1}{2}(\phi_3 + \phi_4) = \pi_1\phi_4$$

Thus, the symmetrical subspace U_1 is three-dimensional. Its basis vectors may be taken as follows:

$$\Phi_1 = \phi_1 + \phi_6 \quad (1a)$$
$$\Phi_2 = \phi_2 + \phi_5 \quad (1b)$$
$$\Phi_3 = \phi_3 + \phi_4 \quad (1c)$$

Subspace U_2

$$\pi_2\phi_1 = \tfrac{1}{2}(E - C_2)\phi_1 = \tfrac{1}{2}(\phi_1 - \phi_6) = -\pi_2\phi_6$$
$$\pi_2\phi_2 = \tfrac{1}{2}(E - C_2)\phi_2 = \tfrac{1}{2}(\phi_2 - \phi_5) = -\pi_2\phi_5$$
$$\pi_2\phi_3 = \tfrac{1}{2}(E - C_2)\phi_3 = \tfrac{1}{2}(\phi_3 - \phi_4) = -\pi_2\phi_4$$

The antisymmetrical subspace U_2 is also three-dimensional. Its basis vectors may be taken as follows

$$\Phi_1 = \phi_1 - \phi_6 \quad (2a)$$
$$\Phi_2 = \phi_2 - \phi_5 \quad (2b)$$
$$\Phi_3 = \phi_3 - \phi_4 \quad (2c)$$

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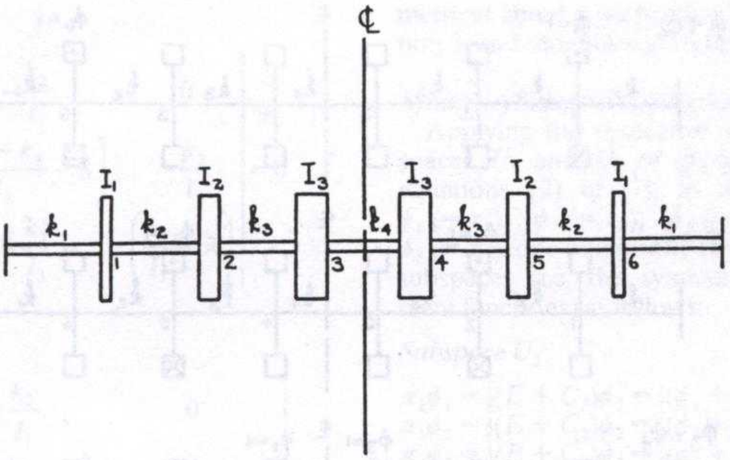

Fig. 1. Shaft–disc torsional system with $n = 6$.

Symmetry-adapted stiffness matrices

Let these be denoted by S_1 and S_2 for subspaces U_1 and U_2, respectively. These matrices will both be 3×3 in size since both U_1 and U_2 are three-dimensional.

To generate the symmetry-adapted stiffness matrices S_1 (for the symmetrical subspace) and S_2 (for the antisymmetrical subspace), unit rotations must be applied upon the system (and the resisting torsional moments noted) in accordance with the ϕ coordinates of the respective basis vectors, as elaborated below.

Each of the sketches in Fig. 2 shows a longitudinal section of the system through the shaft axis, with the shaft itself and the cross-sections of its discs merely shown as lines. A dot in the square at one end of a cross-section of a disc denotes movement (of this end of the disc cross-section) towards the observer, while a cross denotes movement away from the observer—this convention for tangential motion of the disc edges across the plane of the sketches automatically defines the relative senses of the unit rotations associated with a given basis vector.

For either subspace, Φ_1 has components ϕ_1 and ϕ_6, Φ_2 has components ϕ_2 and ϕ_5, and Φ_3 has components ϕ_3 and ϕ_4—see expressions (1) and (2). For the purposes of defining the coefficients of the symmetry-adapted stiffness matrices, disc locations 1 and 6 on the shaft will thus be referred to as the stations of Φ_1. Similarly, locations 2 and 5 will be referred to as the stations of Φ_2, while locations 3 and 4 will be referred to as the stations of Φ_3.

The stiffness coefficients for S_1 and S_2 are obtained in very much the same way as in the conventional procedure. However, instead of the six rotation variables $\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6\}$ of the conventional approach, only three symmetry-adapted rotation functions $\{\Phi_1, \Phi_2, \Phi_3\}$ are now the variables of interest. Also, instead of considering as many as six locations $\{1, 2, 3, 4, 5, 6\}$ in assessing the effects of the unit rotations, the two stations of a given $\Phi_i (i = 1, 2, 3)$ are treated simultaneously, resulting in only three independent locations at which the effects of unit coordinates of a given rotation vector (Φ_1, Φ_2 or Φ_3) are sought.

Thus, for each subspace, the stiffness coefficient s_{ij} ($i = 1, 2, 3; j = 1, 2, 3$) is the value of the moment at any of the (two) stations of Φ_i due to unit rotations at all the stations of Φ_j, while all the discs other than those at the stations of Φ_j are held at rest (see Fig. 2). The senses (clockwise or anticlockwise) of the unit rotations applied at the stations of Φ_j are given by the coefficients (+1 or −1) of the components of Φ_j, as appear in equation (1) (in the case of subspace U_1) or equation (2) (in the case of subspace U_2).

In terms of the torsional constants shown in Fig. 1, the results are as follows:

Subspace U_1

$$s_{11} = k_1 + k_2; \quad s_{12} = -k_2; \quad s_{13} = 0$$
$$s_{21} = -k_2; \quad s_{22} = k_2 + k_3; \quad s_{23} = -k_3$$
$$s_{31} = 0; \quad s_{32} = -k_3; \quad s_{33} = k_3$$

i.e.

$$S_1 = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

Subspace U_2

$$s_{11} = k_1 + k_2; \quad s_{12} = -k_2; \quad s_{13} = 0$$
$$s_{21} = -k_2; \quad s_{22} = k_2 + k_3; \quad s_{23} = -k_3$$
$$s_{31} = 0; \quad s_{32} = -k_3; \quad s_{33} = k_3 + 2k_4$$

i.e.

$$S_2 = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & (k_3 + 2k_4) \end{bmatrix}$$

Eigenvalues

These are obtained from the usual condition [2, 3]

$$|M^{-1}K - \lambda I| = 0 \qquad (3)$$

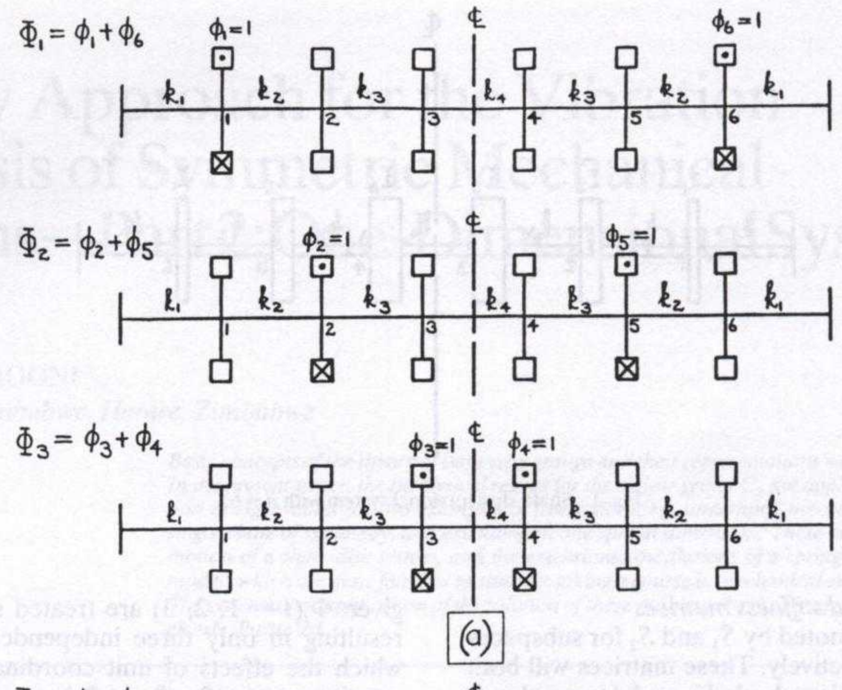

Fig. 2. Unit rotations of discs applied in accordance with the coordinates of the basis vectors for the shaft–disc torsional system: (a) subspace U_1; (b) subspace U_2.

where $\lambda = \omega^2$ and ω is a natural circular frequency of the system. The subspaces are considered independently of each other, so that for subspace U_1, the stiffness matrix K assumes the form S_1, while for subspace U_2, K assumes the form S_2. For either subspace, the non-zero elements of the 3×3 diagonal mass matrix M are simply the mass moments of inertia occurring at (i) any of the two stations of Φ_1, (ii) any of the two stations of Φ_2, and (iii) any of the two stations of Φ_3, i.e.

$$M_1 = M_2 = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

The identity matrix I in equation (3) is, for both subspaces, of dimensions 3×3.

Applying expression (3) to the two subspaces in turn, one obtains the following equations:

Subspace U_1

$$\begin{vmatrix} \left(\frac{k_1+k_2}{I_1}-\lambda\right) & -\frac{k_2}{I_1} & 0 \\ -\frac{k_2}{I_2} & \left(\frac{k_2+k_3}{I_2}-\lambda\right) & -\frac{k_3}{I_2} \\ 0 & -\frac{k_3}{I_3} & \left(\frac{k_3}{I_3}-\lambda\right) \end{vmatrix} = 0$$

Subspace U_2

$$\begin{vmatrix} \left(\frac{k_1+k_2}{I_1}-\lambda\right) & -\frac{k_2}{I_1} & 0 \\ -\frac{k_2}{I_2} & \left(\frac{k_2+k_3}{I_2}-\lambda\right) & -\frac{k_3}{I_2} \\ 0 & -\frac{k_3}{I_3} & \left(\frac{k_3+2k_4}{I_3}-\lambda\right) \end{vmatrix} = 0$$

Expanding the determinant in each of the above equations leads to two independent cubic equations in λ, which, upon solving separately, yield the six required eigenvalues λ_1, λ_2, λ_3, λ_4, λ_5 and λ_6. This represents a considerable simplification in the determination of the eigenvalues, in comparison with the conventional approach that would require the evaluation of a full 6×6 determinant—not a trivial task—and the solution of an ensuing sixth-degree polynomial equation.

A SPRING–MASS RECTILINEAR SYSTEM

The second example involves a spring–mass system with five degrees of freedom, describing the rectilinear motions x_1, x_2, x_3, x_4 and x_5 of masses m_1 (at locations 1 and 5), m_2 (at locations 2 and 4) and m_3 (at location 3), as shown in Fig. 3. The spring constants (k) of the various intervals of the system are also shown in the figure. It is clear that the stationary configuration of the system is symmetrical about a perpendicular axis through location 3, and thus belongs to the group C_2.

Symmetry-adapted displacement functions

Applying the respective idempotents for subspaces U_1 and U_2 of group C_2, as defined by equations (2) of [1], to the system functions $\phi_1 (= x_1)$, $\phi_2 (= x_2)$, $\phi_3 (= x_3)$, $\phi_4 (= x_4)$ and $\phi_5 (= x_5)$, one obtains the basis vectors for the two subspaces (i.e. the symmetry-adapted displacement functions) as follows:

Subspace U_1

$$\pi_1\phi_1 = \tfrac{1}{2}(E + C_2)\phi_1 = \tfrac{1}{2}(\phi_1 + \phi_5) = \pi_1\phi_5$$
$$\pi_1\phi_2 = \tfrac{1}{2}(E + C_2)\phi_2 = \tfrac{1}{2}(\phi_2 + \phi_4) = \pi_1\phi_4$$
$$\pi_1\phi_3 = \tfrac{1}{2}(E + C_2)\phi_3 = \tfrac{1}{2}(\phi_3 + \phi_3) = \phi_3$$

Thus, the symmetrical subspace U_1 is three-dimensional, with basis vectors

$$\Phi_1 = \phi_1 + \phi_5 \quad (4a)$$
$$\Phi_2 = \phi_2 + \phi_4 \quad (4b)$$
$$\Phi_3 = \phi_3 \quad (4c)$$

Subspace U_2

$$\pi_2\phi_1 = \tfrac{1}{2}(E - C_2)\phi_1 = \tfrac{1}{2}(\phi_1 - \phi_5) = -\pi_2\phi_5$$
$$\pi_2\phi_2 = \tfrac{1}{2}(E - C_2)\phi_2 = \tfrac{1}{2}(\phi_2 - \phi_4) = -\pi_2\phi_4$$
$$\pi_2\phi_3 = \tfrac{1}{2}(E - C_2)\phi_3 = \tfrac{1}{2}(\phi_3 - \phi_3) = 0$$

The antisymmetrical subspace U_2 is two-dimensional, with basis vectors

$$\Phi_1 = \phi_1 - \phi_5 \quad (5a)$$
$$\Phi_2 = \phi_2 - \phi_4 \quad (5b)$$

Symmetry-adapted stiffness matrices

For each subspace, these are obtained as explained for the shaft–disc example, with unit displacements now taking the place of unit rotations, and axial forces now taking the place of moments. In working out the stiffness coefficients, the unit displacements of the masses are applied in accordance with the coordinates of the basis vectors (equations 4 or 5), as illustrated in Fig. 4. Thus, for a given subspace of

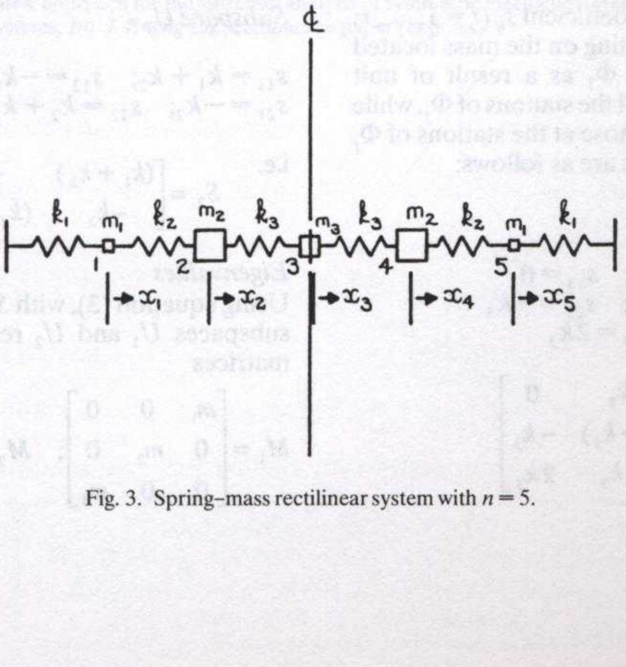

Fig. 3. Spring–mass rectilinear system with $n = 5$.

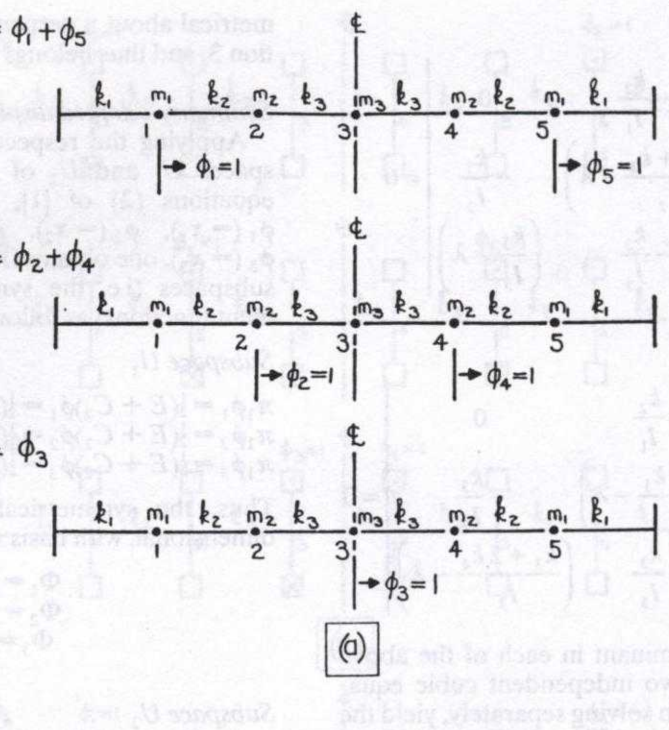

Fig. 4. Unit displacements of masses applied in accordance with the coordinates of the basis vectors for the spring–mass rectilinear system: (a) subspace U_1; (b) subspace U_2.

dimension r, the stiffness coefficient s_{ij} ($i = 1, \ldots, r$; $j = 1, \ldots, r$) is the force acting on the mass located at any of the stations of Φ_i as a result of unit displacements applied at all the stations of Φ_j, while all the masses other than those at the stations of Φ_j are held at rest. The results are as follows:

Subspace U_1

$$s_{11} = k_1 + k_2; \quad s_{12} = -k_2; \quad s_{13} = 0$$
$$s_{21} = -k_2; \quad s_{22} = k_2 + k_3; \quad s_{23} = -k_3$$
$$s_{31} = 0; \quad s_{32} = -2k_3; \quad s_{33} = 2k_3$$

i.e.

$$S_1 = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -2k_3 & 2k_3 \end{bmatrix}$$

Subspace U_2

$$s_{11} = k_1 + k_2; \quad s_{12} = -k_2$$
$$s_{21} = -k_2; \quad s_{22} = k_2 + k_3$$

i.e.

$$S_2 = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix}$$

Eigenvalues

Using equation (3), with S_1 and S_2 in place of K for subspaces U_1 and U_2 respectively, and the mass matrices

$$M_1 = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}; \quad M_2 = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

for subspaces U_1 and U_2, respectively, the following equations are obtained.

Subspace U_1

$$\begin{vmatrix} \left(\dfrac{k_1+k_2}{m_1}-\lambda\right) & -\dfrac{k_2}{m_1} & 0 \\ -\dfrac{k_2}{m_2} & \left(\dfrac{k_2+k_3}{m_2}-\lambda\right) & -\dfrac{k_3}{m_2} \\ 0 & -\dfrac{2k_3}{m_3} & \left(\dfrac{2k_3}{m_3}-\lambda\right) \end{vmatrix} = 0$$

Subspace U_2

$$\begin{vmatrix} \left(\dfrac{k_1+k_2}{m_1}-\lambda\right) & -\dfrac{k_2}{m_1} \\ -\dfrac{k_2}{m_2} & \left(\dfrac{k_2+k_3}{m_2}-\lambda\right) \end{vmatrix} = 0$$

Expanding the determinant in each of the above equations results in two independent equations in λ, one a cubic and the other a quadratic; upon solving these separately, the five required eigenvalues ensue. The conventional method would have required the evaluation of a 5×5 determinant for the full system, and the solution of the resulting fifth-degree polynomial equation.

CONCLUSION

The application of symmetry group C_2 to two examples of one-dimensional mechanical systems has been illustrated. These have comprised a shaft-disc torsional system with an even number of degrees of freedom, and a spring–mass rectilinear system with an odd number of degrees of freedom, both configurations having a single plane of symmetry.

Denoting the total number of degrees of freedom of any C_2 system by n, group-theory decomposition of the problem into the constituent symmetrical and antisymmetrical portions results in two independent polynomial equations each of degree $n/2$ if n is even; if n is odd, the symmetrical subspace yields a polynomial equation of degree $(n+1)/2$, while the antisymmetrical subspace yields a polynomial of degree $(n-1))/2$. In either case, the computational effort involved in solving for the n eigenvalues of the problem on the basis of these two mutually independent subspace polynomials is only a fraction of that which would be incurred were a polynomial equation of full degree n (as yielded by the conventional method) to be tackled.

In the third and final part [4], the illustration of the application of group theory to the vibration of symmetric mechanical systems is extended to the more diverse class of two-dimensional problems, where even greater computational gains can be achieved. More specifically, the theory developed in the first part [1] for symmetry groups C_{2v}, C_{3v} and C_{4v} is applied to the vibration analysis of two-dimensional spring–mass models with symmetries based on the rectangle, the equilateral triangle and the square, respectively.

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