Int. J. Engng Ed. Vol. 12, No. 1, pp. 59–64, 1996
Printed in Great Britain.
0949-149X/91 $3.00+0.00

A New Approach for the Vibration Analysis of Symmetric Mechanical Systems—Part 1: Theoretical Preliminaries

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Group theory can be used to simplify the analysis of a physical system, whenever such a system possesses symmetry properties. In this paper, the first of three parts, a teaching-oriented outline of the general procedure for applying group theory to vibration problems in mechanical engineering is presented for the first time. This is followed by descriptions of the symmetry group applicable to one-dimensional mechanical systems, and three simple groups applicable to two-dimensional systems. For each of these groups, the theoretical results relevant to the practical application of the group to actual engineering problems are summarized. In the second and third articles, the theory is applied to the determination of natural frequencies of vibration for specific examples of one-dimensional and two-dimensional mechanical systems, respectively, with symmetric initial configurations. In this way, the considerable reductions in computational effort achieved through use of the inherently more systematic group-theoretic procedure, instead of the conventional method, are illustrated. The material presented in each one of the three articles can be covered in one double lecture (2 hr) and one tutorial session (2 hr) of a course in structural-mechanical vibrations for senior undergraduates or postgraduates.

EDUCATIONAL SUMMARY

1. The paper discusses materials for a course in structural-mechanical vibrations.
2. Civil engineering and mechanical engineering students are taught in this course.
3. The course is intended for final-year undergraduate and M.Sc. postgraduate students.
4. The course is presented via formal lectures and tutorial sessions.
5. The material is presented in a regular course.
6. The time required to cover the material is two hours formal lectures and two hours tutorial sessions, for each of parts 1, 2 and 3.
7. Student homework or revision hours required for the materials amounts to 1 hr homework for part 1, 1 hr homework for part 2, 2 hr homework for part 3.
8. The use of group theory to decompose eigenvalue problems into simpler, independent subproblems is a novel aspect of this approach.
9. The standard text recommended in the course, in addition to author's notes, is *Vibration of Mechanical and Structural Systems*, by M. L. James, G. M. Smith, J. C. Wolford and P. W. Whaley, Harper & Row, New York (1989).

* Paper accepted 15 May 1995.

INTRODUCTION

THE applicability of group theory to the simplification of the analysis of physical problems involving symmetry is well known [1]. Almost all the applications of the theory have been confined to the domains of physics and chemistry, such as quantum mechanics [2, 3] and molecular symmetry [4, 5]. Zlokovic [6] has pioneered its application to the statics, vibration and stability of civil-engineering structures, by solving specific examples involving beams, grids, simple space frames, a cable net and a plate. The applicability of group theory to structural-engineering computing has been further demonstrated by Zingoni *et al.* [7, 8], who have considered the natural-frequency determination for symmetric configurations of elastic plates on the basis of the finite-difference method [7], and the static analysis of indeterminate space frames by the flexibility method [8]. The author and his colleagues have also recently developed the relevant group-theoretic schemes for the vibration analysis of plane grids [9], and the static analysis of plane frames by the direct stiffness method [10]; a formulation for the linearized static and dynamic analysis of cable nets has also been developed along similar lines [11]. The author's present research efforts are focussed on the development of a general group-theoretic finite-element formulation, utilizing some results which are already available [12].

The simplicity of the procedure for applying group theory to general eigenvalue problems of the vibration and/or stability of symmetrial configurations of mechanical or structural systems, combined with the systematic way in which all the symmetry properties of the system are taken into account (resulting in the maximum possible reduction in computational effort), makes it a particularly suitable topic for incorporating into taught cases on vibration analysis.

The aims of the present three-part contribution are (i) to outline, perhaps for the first time in a mechanical-engineering teaching context, those aspects of group theory that are most relevant to its application to the vibration analysis of symmetric mechanical systems; (ii) to present a step-by-step illustration of the application procedure itself, through a series of very simple examples which students can easily follow; and (iii) to demonstrate the substantial computational advantages of the proposed new procedure over the conventional approach, thereby justifying to lecturers why the topic merits some treatment in advanced courses on vibration of mechanical systems (especially if it is also noted that symmetry features quite frequently in mechanical-engineering systems).

The material should be presented to students of mechanical and/or structural engineering only after the conventional concepts of vibration analysis (i.e. formulation and solution of the eigenvalue problem, natural-frequencies, mode shapes, etc.) have been covered. More specifically, it is intended for senior undergraduates and/or postgraduate students in mechanical engineering. Material in each of the three articles of this contribution can be covered in one formal double lecture (2 hr), and consolidated in one tutorial session (2 hr), implying a total time outlay of 12 hr on this new topic. In teaching the theoretical preliminaries (part 1), it must be noted that group theory is a full course (in mathematics) in its own right, and therefore impossible to cover in depth in any course in engineering; only the important results and consequences need be highlighted for the purpose of teaching the background theory to students of engineering.

OUTLINE OF THE GENERAL PROCEDURE FOR APPLYING GROUP THEORY TO EIGENVALUE PROBLEMS

For a symmetrical configuration of a system with n degrees of freedom, it is possible to adapt the arbitrary displacement functions defining the motion of the system to the symmetry types of independent subspaces (each of dimension a fraction of n) into which the original eigenvalue problem (in an n-dimensional vector space) may be decomposed.

Corresponding to each of these subspaces is a distinct irreducible matrix representation [1] of the symmetry group of the configuration of the system, such a representation being simply a set of the smallest possible transformation matrices for the symmetry elements (i.e. operations) making up the symmetry group. The symmetry-adapted displacement functions for a given subspace are actually generated by applying the corresponding idempotent (which acts as a 'projection operator') [6, 13–15] to each of the n arbitrary displacement functions. These special operators (i.e. idempotents for the various types of irreducible representations), which are mutually orthogonal, can readily be written down from the table of group characters [1, 14, 15]. Character tables for all the symmetry groups that may apply to physical systems, of which mechanical and structural systems form a part, are widely available in the literature (see, for example, [1], [5], [13] or [16]).

Eigenvalues for the problem are then simply obtained by consideration of the symmetry-adapted functions for one subspace at a time, independently of those of the other subspaces. In this way, instead of solving a relatively complex polynomial equation of (large) degree n, one needs only solve, independently of each other, as many considerably simpler polynomial equations (each of degree a fraction of n) as there are subspaces.

THE SYMMETRY GROUPS C_2, C_{2v}, C_{3v} AND C_{4v}

The group C_2, describing the in-plane symmetry of, say, a straight line AB about an axis 1–1 perpendicular to the line and passing through its midpoint O (see Fig. 1a), has two symmetry elements, namely

- E identity
- C_2 180° in-plane rotation about O

which permute the ends A and B of the line. This group is relevant to one-dimensional configurations of systems, which can have no more than single-fold symmetry (i.e. one plane of symmetry).

The groups C_{2v}, C_{3v} and C_{4v}, describing the in-plane symmetries of the rectangle, the equilateral triangle and the square, respectively, as shown in Fig. 1, are only applicable to two-dimensional configurations. It must be noted that two-dimensional systems, unlike one-dimensional ones, can assume symmetrical configurations of an infinite number of symmetry groups (e.g. C_m or C_{mv} where $m = 2, 3, \ldots, \infty$); the groups C_{2v}, C_{3v} and C_{4v}, being among the simplest, and probably the ones most likely to apply to simple models of two-dimensional mechanical systems, have been chosen for illustrative purposes.

The symmetry operations associated with the group C_{2v} (refer to the basic configuration of Fig. 1b) are as follows:

- E identity
- C_2 180° in-plane rotation about the vertical axis through the centre of symmetry O
- σ_x reflection in the vertical plane containing the x axis
- σ_y reflection in the vertical plane containing the y axis

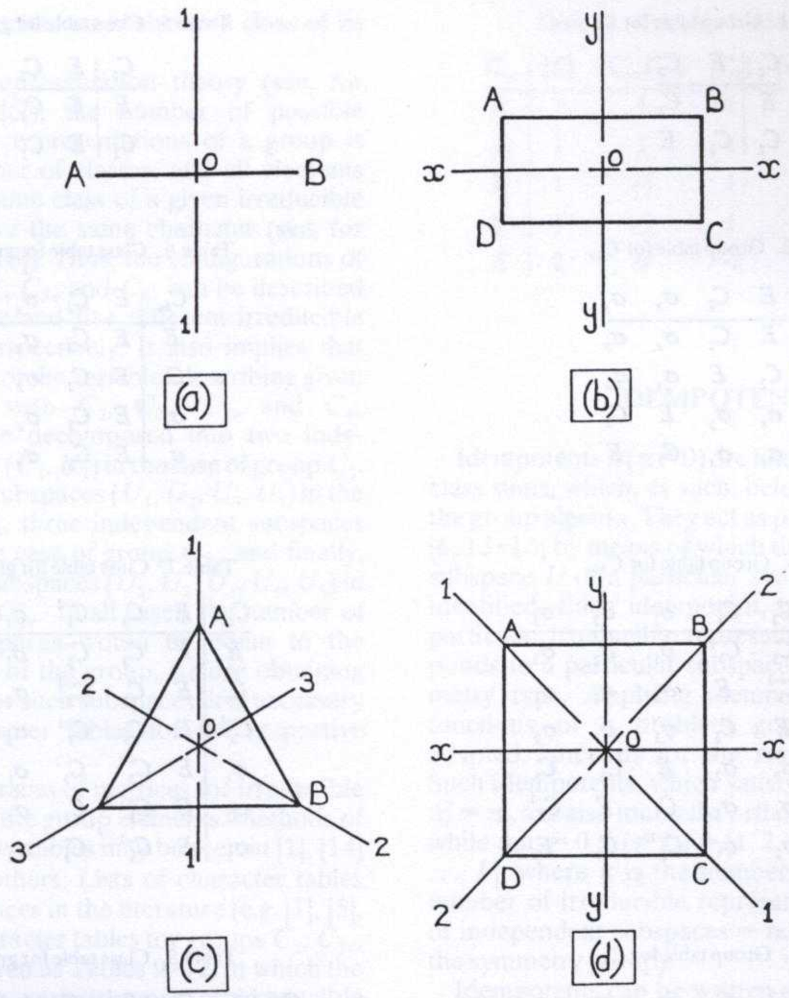

Fig. 1. Symmetry planes and rotation axes for (a) a straight line (C_2), (b) a rectangle (C_{2v}), (c) an equilateral triangle (C_{3v}), and (d) a square (C_{4v}).

In addition to all the above symmetry operations, the group C_{4v}, whose basic configuration is shown in Fig. 1d, also has the following elements:

C_4 90° clockwise rotation about the vertical axis through the centre of symmetry O

C_4^{-1} 90° anticlockwise rotation about the vertical axis through O

σ_1 reflection in the vertical plane containing the diagonal axis 1–1

σ_2 reflection in the vertical plane containing the diagonal axis 2–2

The symmetry operations of groups C_{2v} and C_{4v} permute the respective corners A, B, C and D of the rectangle and the square. These groups are of order 4 and 8, respectively, the order of a group being simply the number of elements (in this case, symmetry operations) making up the group.

As illustrated by the basic configuration of Fig. 1c, the group C_{3v} is of order 6, with symmetry elements

E identity

C_3 120° clockwise rotation about the vertical axis through centre of symmetry O

C_3^{-1} 120° anticlockwise rotation about the vertical axis through O

σ_1 reflection in the vertical plane containing the axis 1–1

σ_2 reflection in the vertical plane containing the axis 2–2

σ_3 reflection in the vertical plane containing the axis 3–3

GROUP, CLASS AND CHARACTER TABLES

Multiplication combinations of group elements generate the group table. In the group tables for C_2, C_{2v}, C_{3v} and C_{4v} given as Tables 1–4, the name of the group appears in the top-left corner. The order of the multiplication is defined by $\alpha\beta = \gamma$, where the element α is taken from the left side and the element β from the top, and their product γ is entered at the intersection of row α and column β. It is noted that entries of a group table are elements of the group itself. Groups C_2 and C_{2v} are described as Abelian because their group tables are symmetrical about the principal diagonal (i.e. group elements multiply commutatively).

Table 1. Group table for C_2.

C_2	E	C_2
E	E	C_2
C_2	C_2	E

Table 2. Group table for C_{2v}.

C_{2v}	E	C_2	σ_x	σ_y
E	E	C_2	σ_x	σ_y
C_2	C_2	E	σ_y	σ_x
σ_x	σ_x	σ_y	E	C_2
σ_y	σ_y	σ_x	C_2	E

Table 3. Group table for C_{3v}.

C_{3v}	E	C_3	C_3^{-1}	σ_1	σ_2	σ_3
E	E	C_3	C_3^{-1}	σ_1	σ_2	σ_3
C_3	C_3	C_3^{-1}	E	σ_2	σ_3	σ_1
C_3^{-1}	C_3^{-1}	E	C_3	σ_3	σ_1	σ_2
σ_1	σ_1	σ_3	σ_2	E	C_3^{-1}	C_3
σ_2	σ_2	σ_1	σ_3	C_3	E	C_3^{-1}
σ_3	σ_3	σ_2	σ_1	C_3^{-1}	C_3	E

Table 4. Group table for C_{4v}.

C_{4v}	E	C_4	C_4^{-1}	C_2	σ_x	σ_y	σ_1	σ_2
E	E	C_4	C_4^{-1}	C_2	σ_x	σ_y	σ_1	σ_2
C_4	C_4	C_2	E	C_4^{-1}	σ_1	σ_2	σ_y	σ_x
C_4^{-1}	C_4^{-1}	E	C_2	C_4	σ_2	σ_1	σ_x	σ_y
C_2	C_2	C_4^{-1}	C_4	E	σ_y	σ_x	σ_2	σ_1
σ_x	σ_x	σ_2	σ_1	σ_y	E	C_2	C_4^{-1}	C_4
σ_y	σ_y	σ_1	σ_2	σ_x	C_2	E	C_4	C_4^{-1}
σ_1	σ_1	σ_x	σ_y	σ_2	C_4	C_4^{-1}	E	C_2
σ_2	σ_2	σ_y	σ_x	σ_1	C_4^{-1}	C_4	C_2	E

Table 5. Class table for group C_2.

C_2	E	C_2
E	E	C_2
C_2	E	C_2

Table 6. Class table for group C_{2v}.

C_{2v}	E	C_2	σ_x	σ_y
E	E	C_2	σ_x	σ_y
C_2	E	C_2	σ_x	σ_y
σ_x	E	C_2	σ_x	σ_y
σ_y	E	C_2	σ_x	σ_y

Table 7. Class table for group C_{3v}.

C_{3v}	E	C_3	C_3^{-1}	σ_1	σ_2	σ_3
E	E	C_3	C_3^{-1}	σ_1	σ_2	σ_3
C_3	E	C_3	C_3^{-1}	σ_2	σ_3	σ_1
C_3^{-1}	E	C_3	C_3^{-1}	σ_3	σ_1	σ_2
σ_1	E	C_3^{-1}	C_3	σ_1	σ_3	σ_2
σ_2	E	C_3^{-1}	C_3	σ_3	σ_2	σ_1
σ_3	E	C_3^{-1}	C_3	σ_2	σ_1	σ_3

Table 8. Class table for group C_{4v}.

C_{4v}	E	C_4	C_4^{-1}	C_2	σ_x	σ_y	σ_1	σ_2
E	E	C_4	C_4^{-1}	C_2	σ_x	σ_y	σ_1	σ_2
C_4	E	C_4	C_4^{-1}	C_2	σ_y	σ_x	σ_2	σ_1
C_4^{-1}	E	C_4	C_4^{-1}	C_2	σ_y	σ_x	σ_2	σ_1
C_2	E	C_4	C_4^{-1}	C_2	σ_x	σ_y	σ_1	σ_2
σ_x	E	C_4^{-1}	C_4	C_2	σ_x	σ_y	σ_2	σ_1
σ_y	E	C_4^{-1}	C_4	C_2	σ_x	σ_y	σ_2	σ_1
σ_1	E	C_4^{-1}	C_4	C_2	σ_y	σ_x	σ_1	σ_2
σ_2	E	C_4^{-1}	C_4	C_2	σ_y	σ_x	σ_1	σ_2

Classes of a group are obtained by first evaluating $\alpha^{-1}\beta\alpha$ for all elements α and β of the group; taking α from the left side and β from the top, class tables for groups C_2, C_{2v}, C_{3v} and C_{4v} are obtained, with the help of the associated multiplication tables, as given in Tables 5–8.

The various classes are then the distinct sets of symmetry elements formed by collecting into a set, for each symmetry element β of the group, the distinct results of the conjugates $\alpha^{-1}\beta\alpha$ for all elements α of the group. Thus, group C_2 has two classes with one element each, namely

$$K_1 = \{E\}; \quad K_2 = \{C_2\}$$

while group C_{2v} has four classes, each also with only one element:

$$K_1 = \{E\}; \quad K_2 = \{C_2\}; \quad K_3 = \{\sigma_x\}; \quad K_4 = \{\sigma_y\}$$

Similarly, the following classes are readily obtained for groups C_{3v} and C_{4v}.

$$C_{3v}: \quad K_1 = \{E\}; \quad K_2 = \{C_3, C_3^{-1}\}; \quad K_3 = \{\sigma_1, \sigma_2, \sigma_3\}$$

$$C_{4v}: \quad K_1 = \{E\}; \quad K_2 = \{C_4, C_4^{-1}\}; \quad K_3 = \{C_2\}; \quad K_4 = \{\sigma_x, \sigma_y\}; \quad K_5 = \{\sigma_1, \sigma_2\}$$

Notice that the identity element E always belongs to a class of its own, and that for the Abelian groups

Table 1. Group table for C_2.

C_2	E	C_2
E	E	C_2
C_2	C_2	E

Table 2. Group table for C_{2v}.

C_{2v}	E	C_2	σ_x	σ_y
E	E	C_2	σ_x	σ_y
C_2	C_2	E	σ_y	σ_x
σ_x	σ_x	σ_y	E	C_2
σ_y	σ_y	σ_x	C_2	E

Table 3. Group table for C_{3v}.

C_{3v}	E	C_3	C_3^{-1}	σ_1	σ_2	σ_3
E	E	C_3	C_3^{-1}	σ_1	σ_2	σ_3
C_3	C_3	C_3^{-1}	E	σ_2	σ_3	σ_1
C_3^{-1}	C_3^{-1}	E	C_3	σ_3	σ_1	σ_2
σ_1	σ_1	σ_3	σ_2	E	C_3^{-1}	C_3
σ_2	σ_2	σ_1	σ_3	C_3	E	C_3^{-1}
σ_3	σ_3	σ_2	σ_1	C_3^{-1}	C_3	E

Table 4. Group table for C_{4v}.

C_{4v}	E	C_4	C_4^{-1}	C_2	σ_x	σ_y	σ_1	σ_2
E	E	C_4	C_4^{-1}	C_2	σ_x	σ_y	σ_1	σ_2
C_4	C_4	C_2	E	C_4^{-1}	σ_1	σ_2	σ_y	σ_x
C_4^{-1}	C_4^{-1}	E	C_2	C_4	σ_2	σ_1	σ_x	σ_y
C_2	C_2	C_4^{-1}	C_4	E	σ_y	σ_x	σ_2	σ_1
σ_x	σ_x	σ_2	σ_1	σ_y	E	C_2	C_4^{-1}	C_4
σ_y	σ_y	σ_1	σ_2	σ_x	C_2	E	C_4	C_4^{-1}
σ_1	σ_1	σ_x	σ_y	σ_2	C_4	C_4^{-1}	E	C_2
σ_2	σ_2	σ_y	σ_x	σ_1	C_4^{-1}	C_4	C_2	E

Table 5. Class table for group C_2.

C_2	E	C_2
E	E	C_2
C_2	E	C_2

Table 6. Class table for group C_{2v}.

C_{2v}	E	C_2	σ_x	σ_y
E	E	C_2	σ_x	σ_y
C_2	E	C_2	σ_x	σ_y
σ_x	E	C_2	σ_x	σ_y
σ_y	E	C_2	σ_x	σ_y

Table 7. Class table for group C_{3v}.

C_{3v}	E	C_3	C_3^{-1}	σ_1	σ_2	σ_3
E	E	C_3	C_3^{-1}	σ_1	σ_2	σ_3
C_3	E	C_3	C_3^{-1}	σ_2	σ_3	σ_1
C_3^{-1}	E	C_3	C_3^{-1}	σ_3	σ_1	σ_2
σ_1	E	C_3^{-1}	C_3	σ_1	σ_3	σ_2
σ_2	E	C_3^{-1}	C_3	σ_3	σ_2	σ_1
σ_3	E	C_3^{-1}	C_3	σ_2	σ_1	σ_3

Table 8. Class table for group C_{4v}.

C_{4v}	E	C_4	C_4^{-1}	C_2	σ_x	σ_y	σ_1	σ_2
E	E	C_4	C_4^{-1}	C_2	σ_x	σ_y	σ_1	σ_2
C_4	E	C_4	C_4^{-1}	C_2	σ_y	σ_x	σ_2	σ_1
C_4^{-1}	E	C_4	C_4^{-1}	C_2	σ_y	σ_x	σ_2	σ_1
C_2	E	C_4	C_4^{-1}	C_2	σ_x	σ_y	σ_1	σ_2
σ_x	E	C_4^{-1}	C_4	C_2	σ_x	σ_y	σ_2	σ_1
σ_y	E	C_4^{-1}	C_4	C_2	σ_x	σ_y	σ_2	σ_1
σ_1	E	C_4^{-1}	C_4	C_2	σ_y	σ_x	σ_1	σ_2
σ_2	E	C_4^{-1}	C_4	C_2	σ_y	σ_x	σ_1	σ_2

Classes of a group are obtained by first evaluating $\alpha^{-1}\beta\alpha$ for all elements α and β of the group; taking α from the left side and β from the top, class tables for groups C_2, C_{2v}, C_{3v} and C_{4v} are obtained, with the help of the associated multiplication tables, as given in Tables 5–8.

The various classes are then the distinct sets of symmetry elements formed by collecting into a set, for each symmetry element β of the group, the distinct results of the conjugates $\alpha^{-1}\beta\alpha$ for all elements α of the group. Thus, group C_2 has two classes with one element each, namely

$$K_1 = \{E\}; \quad K_2 = \{C_2\}$$

while group C_{2v} has four classes, each also with only one element:

$$K_1 = \{E\}; \quad K_2 = \{C_2\}; \quad K_3 = \{\sigma_x\}; \quad K_4 = \{\sigma_y\}$$

Similarly, the following classes are readily obtained for groups C_{3v} and C_{4v}:

$$C_{3v}: \quad K_1 = \{E\}; \quad K_2 = \{C_3, C_3^{-1}\}; \quad K_3 = \{\sigma_1, \sigma_2, \sigma_3\}$$

$$C_{4v}: \quad K_1 = \{E\}; \quad K_2 = \{C_4, C_4^{-1}\}; \quad K_3 = \{C_2\}; \quad K_4 = \{\sigma_x, \sigma_y\}; \quad K_5 = \{\sigma_1, \sigma_2\}$$

Notice that the identity element E always belongs to a class of its own, and that for the Abelian groups

C_2 and C_{2v}, every element is also in a class of its own.

According to representation theory (see, for example, [1] or [6]), the number of possible irreducible matrix representations of a group is equal to the number of classes, and all elements belonging to the same class of a given irreducible representation have the same character (see, for example, [6] and [13]). Thus, the configurations of the groups C_2, C_{2v}, C_{3v} and C_{4v} can be described by two, four, three and five different irreducible representations, respectively. It also implies that the vector space V of the variables describing given physical systems with C_2, C_{2v}, C_{3v} and C_{4v} symmetries can be decomposed into two independent subspaces (U_1, U_2) in the case of group C_2, four independent subspaces (U_1, U_2, U_3, U_4) in the case of group C_{2v}, three independent subspaces (U_1, U_2, U_3) in the case of group C_{3v}, and finally, five independent subspaces (U_1, U_2, U_3, U_4, U_5) in the case of group C_{4v}. In all cases, the number of independent subspaces would be equal to the number of classes of the group. Before obtaining the basis vectors for such subspaces, it is necessary to have the character tables for the respective groups.

Characters are traces of matrices for irreducible representations of the group elements. Methods of generating character tables may be seen in [1], [14] and [15], among others. Lists of character tables appear in many places in the literature (e.g. [1], [5], [13] and [16]). Character tables for groups C_2, C_{2v}, C_{3v} and C_{4v} are given as Tables 9–12, in which the left side gives the various types of irreducible representations of the group (in standard notation), and the top gives the classes of the group; it must be noted that elements belonging to the same class all have the same character.

Table 9. Character table for group C_2.

C_2	$\{E\}$	$\{C_2\}$
A	1	1
B	1	−1

Table 10. Character table for group C_{2v}.

C_{2v}	$\{E\}$	$\{C_2\}$	$\{\sigma_x\}$	$\{\sigma_y\}$
A_1	1	1	1	1
A_2	1	1	−1	−1
B_1	1	−1	1	−1
B_2	1	−1	−1	1

Table 11. Character table for group C_{3v}.

C_{3v}	$\{E\}$	$\{C_3, C_3^{-1}\}$	$\{\sigma_1, \sigma_2, \sigma_3\}$
A_1	1	1	1
A_2	1	1	−1
E	2	−1	0

Table 12. Character table for group C_{4v}.

C_{4v}	$\{E\}$	$\{C_4, C_4^{-1}\}$	$\{C_2\}$	$\{\sigma_x, \sigma_y\}$	$\{\sigma_1, \sigma_2\}$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
B_1	1	−1	1	1	−1
B_2	1	−1	1	−1	1
E	2	0	−2	0	0

IDEMPOTENTS

Idempotents $\pi(\pi \neq 0)$ are linear combinations of class sums, which, as such, belong to the centre of the group algebra. They act as projection operators [6, 13–15] by means of which the basis vectors of a subspace U of a particular symmetry type can be identified. Each idempotent, corresponding to a particular irreducible representation, also corresponds to a particular subspace of a specific symmetry type. Applying idempotents to arbitrary functions of a problem generates symmetry-adapted functions for the respective subspaces. Such idempotents, which satisfy the usual relation $\pi_i^2 = \pi_i$, are also mutually orthogonal (i.e. $\pi_i\pi_i \neq 0$, while $\pi_i\pi_j = 0$ if $i \neq j$; $i = \{1, 2, .., k\}$ and $j = \{1, 2, \ldots, k\}$, where k is the number of idempotents = number of irreducible representations = number of independent subspaces = number of classes of the symmetry group).

Idempotents can be written down directly from the character table, using the expression [6]

$$\pi_i = \frac{h_i}{h} \sum_{\alpha} \chi_i(\alpha^{-1})\alpha \qquad (1)$$

where h_i is the dimension of the ith irreducible representation, given by $\chi_i(E)$, the first value of the ith row of the character table; h is the order of the symmetry group (i.e. the number of elements in the group); χ_i is a character of the ith irreducible representation; α is a symmetry element of the group, and α^{-1} its inverse.

Noting that $E^{-1} = E$ and $C_2^{-1} = C_2$, the orthogonal idempotents of group C_2, corresponding to subspaces U_1 and U_2 of the group, respectively, are thus

$$\pi_1 = \tfrac{1}{2}(E + C_2) \qquad (2a)$$
$$\pi_2 = \tfrac{1}{2}(E - C_2) \qquad (2b)$$

The basis vectors of the two subspaces U_1 and U_2 associated with group C_2 are obtained by applying π_1 (in the case of U_1) and π_2 (in the case of U_2) to all the displacement functions $\{\phi_1, \phi_2, \ldots, \phi_n\}$ describing the motion of a system with n degrees of freedom.

Similarly, noting that $\sigma^{-1} = \sigma$, the idempotents for groups C_{2v}, C_{3v} and C_{4v} are obtained from equation (1) as follows (with idempotent π_i corresponding to subspace U_i of a given problem):

Group C_{2v}

$$\pi_1 = \tfrac{1}{4}(E + C_2 + \sigma_x + \sigma_y) \quad (3a)$$
$$\pi_2 = \tfrac{1}{4}(E + C_2 - \sigma_x - \sigma_y) \quad (3b)$$
$$\pi_3 = \tfrac{1}{4}(E - C_2 + \sigma_x - \sigma_y) \quad (3c)$$
$$\pi_4 = \tfrac{1}{4}(E - C_2 - \sigma_x + \sigma_y) \quad (3d)$$

Group C_{3v}

$$\pi_1 = \tfrac{1}{6}(E + C_3 + C_3^{-1} + \sigma_1 + \sigma_2 + \sigma_3) \quad (4a)$$
$$\pi_2 = \tfrac{1}{6}(E + C_3 + C_3^{-1} - \sigma_1 - \sigma_2 - \sigma_3) \quad (4b)$$
$$\pi_3 = \tfrac{2}{6}(2E - C_3 - C_3^{-1}) = \tfrac{1}{3}(2E - C_3 - C_3^{-1}) \quad (4c)$$

Group C_{4v}

$$\pi_1 = \tfrac{1}{8}(E + C_4 + C_4^{-1} + C_2 + \sigma_x + \sigma_y + \sigma_1 + \sigma_2) \quad (5a)$$
$$\pi_2 = \tfrac{1}{8}(E + C_4 + C_4^{-1} + C_2 - \sigma_x - \sigma_y - \sigma_1 - \sigma_2) \quad (5b)$$
$$\pi_3 = \tfrac{1}{8}(E - C_4 - C_4^{-1} + C_2 + \sigma_x + \sigma_y - \sigma_1 - \sigma_2) \quad (5c)$$
$$\pi_4 = \tfrac{1}{8}(E - C_4 - C_4^{-1} + C_2 - \sigma_x - \sigma_y + \sigma_1 + \sigma_2) \quad (5d)$$
$$\pi_5 = \tfrac{2}{8}(2E - 2C_2) = \tfrac{1}{2}(E - C_2) \quad (5e)$$

As with group C_2, the basis vectors of the four subspaces associated with the group C_{2v}, the three subspaces associated with the group C_{3v}, and the five subspaces associated with the group C_{4v}, are obtained by applying the respective idempotents to all the functions $\{\phi_1, \phi_2, \ldots, \phi_n\}$ describing the motion of a system with n degrees of freedom.

CONCLUSION

In this article, the aspects of group theory most relevant to its application to the determination of eigenvalues for symmetric mechanical systems have been outlined.

For engineering students, an understanding of the mathematical proofs underlying some of the statements made in the foregoing account is not essential. Instead, emphasis should be placed on the actual procedure for solving problems, which will be illustrated in the follow-up articles [17, 18].

In parts 2 and 3 [17, 18], the idempotents derived in the present paper will be used to generate basis vectors (i.e. symmetry-adapted functions) for the various subspaces of specific examples, on the basis of which the entire set of eigenvalues for a given system may be obtained by considering its subspaces independently of each other. In other words, instead of solving a polynomial of full degree n for the n roots (i.e. eigenvalues) of the system, it will only be necessary to solve, one at a time, a small number of polynomials whose individual degrees will only be a fraction of n, to obtain the required eigenvalues for the entire system.

One-dimensional mechanical systems of C_2 configuration will be considered in the next part [17], while two-dimensional systems with C_{2v}, C_{3v} and C_{4v} configurations will form the subject of the third and final article [18].

REFERENCES

1. M. Hamermesh, *Group Theory and its Application to Physical Problems*, Pergamon Press, Oxford (1962).
2. E. P. Wigner, *Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra*, Academic Press, New York (1959).
3. M. Tinkham, *Group Theory and Quantum Mechanics*, McGraw-Hill, New York (1975).
4. L. H. Hall, *Group Theory and Symmetry in Chemistry*, McGraw-Hill, New York (1969).
5. D. Schonland, *Molecular Symmetry*. Van Nostrand, New York (1965).
6. G. M. Zlokovic, *Group Theory and G-vector Spaces in Structural Analysis*, Ellis Horwood, Chichester (1989).
7. A. Zingoni, M. N. Pavlovic, D. Lloyd-Smith and G. M. Zlokovic, Group-theory considerations of finite-difference plate eigenvalue problems. *Proc. 5th Int. Conference on Civil and Structural Engineering Computing*, Vol. 6: *Developments in Computational Engineering Mechanics*, Civil-Comp Press, Edinburgh, pp. 243–256 (1993).
8. A. Zingoni, M. N. Pavlovic, D. Lloyd-Smith and G. M. Zlokovic, Application of group theory to the analysis of space frames. *Proc. 4th Int. Conference on Space Structures*, Thomas Telford, London, pp. 1334–1347 (1993).
9. A. Zingoni and M. N. Pavlovic, On natural-frequency determination of symmetric grid-mass systems. *Proc. 5th Int. Conference on Recent Advances in Structural Dynamics*, Institute of Sound and Vibration Research, Southampton, pp. 151–163 (1994).
10. A. Zingoni, M. N. Pavlovic and G. M. Zlokovic, Symmetry and the direct stiffness method in structural analysis: a formulation based on group theory. *Proc. 2nd Int. Conference on Computational Structures Technology, Vol. G: Advances in Computational Mechanics*, Civil-Comp Press, Edinburgh, pp. 107–115 (1994).
11. A. Zingoni, An efficient computational scheme for the vibration analysis of high-tension cable nets, *J. Sound Vibration*, Vol. 189 (No. 1), pp. 55–79 (1996).
12. G. M. Zlokovic, *Group Supermatrices in Finite Element Analysis*, Ellis Horwood, Chichester (1992).
13. J. W. Leech and D. J. Newman, *How to Use Groups*. Methuen, London (1969).
14. N. V. V. J. Swamy and M. A. Samuel, *Group Theory Made Easy for Scientists and Engineers*, Wiley-Interscience, New York (1979).
15. G. G. Hall, *Applied Group Theory*, Longman, London (1967).
16. A. P. Cracknell, *Applied Group Theory*, Pergamon Press, Oxford (1968).
17. A. Zingoni, A new approach for the vibration analysis of symmetric mechanical systems—part 2: one-dimensional systems, *Int. J. Appl. Engng Ed.*, companion paper.
18. A. Zingoni, A new approach for the vibration analysis of symmetric mechanical systems—part 3: two-dimensional systems, *Int. J. Appl. Engng Ed.*, companion paper.