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Maximum-power Efficiency of a Carnot Engine*

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The Curzon–Ahlborn efficiency of a Carnot engine at maximum power output, $\eta^* = 1 - \sqrt{T_1/T_2}$, is graphically derived by maximizing the area of a rectangle in a diagram. By use of proper symmetry the problem can be recast in terms of finding an associated rectangle of given area T_1T_2 and minimal circumference. This provides a geometric illustration of the optimal equivalent temperature $T_2^* = \sqrt{T_1T_2}$ as the geometric mean of the reservoir temperatures and the consequent appearance of the square root in the Curzon–Ahlborn formula.

1. The paper discusses material which can be used in the following courses:
   Heat engines, finite-time thermodynamics.
2. Students of the following departments may benefit from the course/discussion in the paper:
   Mechanical engineering, physics.
3. Level of the course:
   Senior or introductory graduate level.
4. Modes of presentation:
   Traditional lecture, or assigned reading, or student presentation.
5. Is the material presented in a regular or elective course?
   The material should be presented in a required course on thermodynamics.
6. Class hours required to cover the material.
   1 hour.
7. Student homework and revision hours required for the material:
   2 hours.
8. Brief description of novel aspects of the paper:
   Despite growing importance of finite-time thermodynamics in the literature, irreversible heat engines, i.e. heat engines with actual power output, are hardly covered in current thermodynamics courses. The irreversible Carnot engine, in comparison with the reversible one, provides an excellent example for illustrating typical aspects that arise from irreversibility. The few texts that derive the maximum-power efficiency use standard procedures of calculus. The ensuing algebra is cumbersome and the square-root term in the final formula comes as somewhat of a surprise. The present graphical derivation is transparent and gives a physical interpretation of the square-root term in the efficiency formula.
9. The standard text recommended in addition to the author's notes:
   El-Wakil [4], Callen [5].
10. Is the material covered in the text? In what way is the text discussion different from the paper?
    The calculus-based derivation in the texts by El-Wakil and Callen is simple in principle but tedious in execution. This paper derives the results directly from graphical representations of energy and entropy.

CURZON–AHLBORN ENGINE

A REVERSIBLE Carnot engine, operating between reservoirs at high temperature T_2 and low temperature T_1, has an efficiency

$$\eta = W/Q_2 = 1 - T_1/T_2 \tag{1}$$

where Q_2 is the heat inflow at T_2, and W the work output of a cycle. However, because of the infinite period τ of a reversible cycle, proceeding quasi-statically through a sequence of equilibrium states, the engine's power

$$p = W/\tau \tag{2}$$

is zero.

In order to address the efficiency of finite-period cycles an irreversible Carnot engine has been analyzed and the efficiency for maximum-power output has been derived. There have actually been two independent derivations—one by Novikov in 1958 for the engineering community [1] and one by Curzon and Ahlborn in 1975 for the physics community [2]—both leading to the same result. Apparently there seems to be so little communication between the engineering and physics communities that the duplication was not noticed until 1994 [3]. One reason why theories of irreversible heat engines are not more widely known is their lack of coverage in thermodynamics texts. Notable

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exceptions are the texts by El-Wakil [4] and Callen [5].

An analysis of an irreversible (or endoreversible) Carnot engine invokes four temperatures. These are the previous high and low reservoir temperatures, T_2 and T_1, and additional high and low operating temperatures, $\tilde{T}_2 < T_2$ and $\tilde{T}_1 > T_1$, for isothermal expansion and compression of the endoreversible engine, respectively. The temperature differences between the engine and the reservoirs give rise to heat fluxes

$$\tilde{Q}_j/t_j = k_j(T_j - \tilde{T}_j) \qquad j = 2, 1 \tag{3}$$

in finite times t_j and with heat-transfer coefficients k_j depending on the thermal conductivity and dimensions of the wall between engine and reservoir.

The cycle period is the sum of the isothermal and adiabatic process times:

$$\tau = t_2 + t_{ad}^{21} + t_1 + t_{ad}^{12} = (t_2 + t_1)(1 + r) \tag{4}$$

Curzon and Ahlborn assume that the ratio r between the durations of the total adiabatic and isothermal processes is constant for given reservoir temperatures. Taking into account thermodynamic sign convention, the (average) power of a finite cycle is given by:

$$\tilde{p} = \tilde{W}/\tau = (\tilde{Q}_2 + \tilde{Q}_1)/\tau) \tag{5}$$

Inserting Eqs (3) and (4) into Eq. (5) gives the power as function of the reservoir and operating temperatures, $\tilde{p} = \tilde{p}(T_2, \tilde{T}_2, \tilde{T}_1, T_1)$. Maximization with respect to both operating temperatures, $\partial\tilde{p}/\partial\tilde{T}_j = 0$, gives, after much algebra, the maximum power $p^*(T_2, T_1)$ as a function of the fixed reservoir temperatures. The efficiency at maximum-power output, frequently called the Curzon–Ahlborn efficiency, comes out as:

$$\eta^* = W^*/Q_2 = 1 - \sqrt{T_1/T_2} \tag{6}$$

The formula of the Curzon–Ahlborn efficiency differs from that of the Carnot efficiency, Eq. (1), by the appearance of a square root. The square-root term is rather surprising and its origin is difficult to trace through the maximization procedure.

EQUIVALENT CYCLE

The seminal work of Novikov [1] and Curzon and Ahlborn [2] has been extended by several authors [6–12]. Independently of Novikov, Rubin [7] and later Chen and Yan [11] employed an equivalent, endoreversible cycle for a finite-time Carnot engine and defined an equivalent temperature T_2' relating the heat inflow and cycle period to the high reservoir temperature and equivalent temperature as

$$Q_2/\tau = k(T_2 - T_2') \tag{7}$$

The coefficient k is a combination of the actual heat-transfer coefficients from Eq. (3), i.e. $k = k(k_2, k_1) = k_2 k_1/(\sqrt{k_2} + \sqrt{k_1})^2$. The equivalent cycle has a heat outflow Q_1' at the low reservoir temperature determined by endoreversibility:

$$Q_2/T_2' + Q_1'/T_1 = 0 \tag{8}$$

and, by conservation of energy, a work output

$$W' = Q_2 + Q_1' \tag{9}$$

With Eqs (8) and (9), the efficiency of the equivalent cycle is, for given temperatures T_2, T_2' and T_1:

$$\eta' = W'/Q_2 = 1 - T_1/T_2' \tag{10}$$

Using Eq. (7), the power of the equivalent cycle, $p' = W'/\tau$, becomes:

$$p' = W'(T_2 - T_2')k/Q_2 \tag{11}$$

and, after eliminating W' with Eq. (9) and Q_1'/Q_2 with Eq. (8):

$$p' = k(1 - T_1/T_2')(T_2 - T_2') \tag{12}$$

The advantage of the equivalent cycle is that, for fixed reservoir temperatures, the cycle power depends only on one variable, $p' = p'(T_2, T_2', T_1)$, instead of depending on two variables as in the original Curzon–Ahlborn expression, $\tilde{p} = \tilde{p}(T_2, \tilde{T}_2, \tilde{T}_1, T_1)$. This enormously simplifies the maximization, $dp'/dT_2' = 0$, at an optimal equivalent temperature T_2^*. A short calculation gives $T_2^* = \sqrt{T_1 T_2}$ as the geometric mean of the reservoir temperatures and, upon inserting in Eq. (10), the Curzon–Ahlborn efficiency, Eq. (6). The maximum power output is $p^* = k(\sqrt{T_2} - \sqrt{T_1})^2$.

DIAGRAMS

A few years ago a diagram was introduced that shows heat flows Q_2 and Q_1 and work W of a reversible Carnot cycle in geometric proportions. As was noticed recently [3], the diagram has also been introduced twice: by Bejan in 1977 for the engineering community [13–15], and by Bucher in 1986 and Wallingford in 1989 for the physics community [16, 17]. The diagram is shown in Fig. 1. A horizontal bar at T_2 represents Q_2 and a bar at T_1 represents both Q_1 and W. The equal length of the top and bottom bar illustrates conservation of energy. The slope, $|Q_2|/T_2 = |Q_1|/T_1$, of the solid slanted line, dividing the bottom bar into Q_1 and W, illustrates constancy of entropy in the cycle.

The diagram can be extended to account for Carnot cycles with irreversibilities. If irreversible processes increase the heat outflow from Q_1 to Q_1' at the expense of the work output, which decreases by the same amount (from W to W'), then this is illustrated in Fig. 1 by a dashed horizontal bar of length Q_2 at an equivalent temperature T_2' such that a slanted (dashed) line from the right edge of the dashed bar to $T = 0$ on the temperature axis divides the bottom bar into Q_1' and W' segments. Because of the underlying concept of endorever-

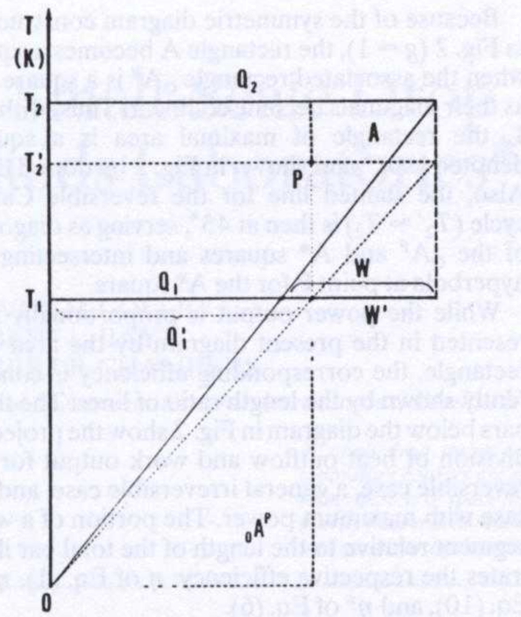

Fig. 1. Diagram for heat flows Q and work W for a reversible (solid lines) and irreversible (dashed lines and primed quantities) Carnot engine. The area A is a proportional measure for the power output. The area $_OA^P$ of the associated rectangle, spanned by origin O and point P, is indicated.

sibility, Eq. (8), the equivalent temperature T_2' in Fig. 1 for, so far unspecified, irreversible processes also represents T_2' of Rubin's and Chen and Yan's equivalent cycle of a Curzon–Ahlborn engine.

A line straight up in Fig. 1 from the intersection of the slanted dashed line with the bottom bar intersects the dashed bar at point P. The top right corner C of the diagram and point P define a rectangle of area $A = W'(T_2 - T_2')$. Comparison with Eq. (11) shows that, for given heat inflow Q_2 and conductivity coefficient k, A is a proportional measure for the power output p'. Thus the objective of maximizing p' amounts to maximizing the area A in the diagram by varying T_2'. Qualitatively, the existence of an optimization problem for the power (area of the rectangle) becomes apparent from the counteracting influence of decreasing work output W' (width of the rectangle) and increasing cycle frequency, $\tau^{-1} \propto (T_2 - T_2')$ due to Eq. (7) (height of the rectangle), as T_2' decreases.

If we regard the diagram in Fig. 1 as a coordinate system with origin O at $T = 0$ on the temperature axis, then the point $P = (x, y)$ has coordinates $x = Q_1'$ and $y = T_2'$. Using a diagram scaling factor $g = Q_2/T_2$, we can express, with Eq. (8), $x = gT_2T_1/T_2'$. We eliminate T_2'. and find that point P lies on a hyperbola, $y(x) = gT_1T_2/x$.

We now consider an associated rectangle, spanned by the origin O and point P. As a basic property of a hyperbola, the area of the associated rectangle, $_OA^P = xy = gT_1T_2$, is constant for any position of P on the hyperbola. Our original task of maximizing the area A can be related to a familiar extreme-value property of the associated rectangle $_OA^P$ if we scale the diagram such that the length of the top bar is equal to the length of T_2 on the temperature axis, i.e. with a scaling factor $g = 1$ as in Fig. 2. In this symmetric situation, a maximization of A is equivalent to finding the associated rectangle of area $_OA^P = xy = T_1T_2$ which has the minimal circumference. It is well known that this is a square with edge length $\sqrt{T_1T_2} = T_2^*$. This provides a geometric illustration for T_2^* as the geometric mean of the reservoir temperatures and, upon inserting into Eq. (10), for the appearance of a square root in the Curzon–Ahlborn formula, Eq. (6).

Two graphical methods related to the Curzon–Ahlborn efficiency have been proposed in recent years. Bejan's method [10, p. 1214] is a geometric construction of the optimum temperature ratio on the absolute temperature scale. The method of Yan and Chen [18] is a graphical derivation of the Curzon–Ahlborn efficiency. The main difference between Yan and Chen's derivation and the present one is that these authors represent rates of heat flow, $q_j = Q_j/\tau$, and the power, $p = W/\tau$, by the lengths of horizontal diagram lines instead of by areas. Expressed in terms of the present notation,

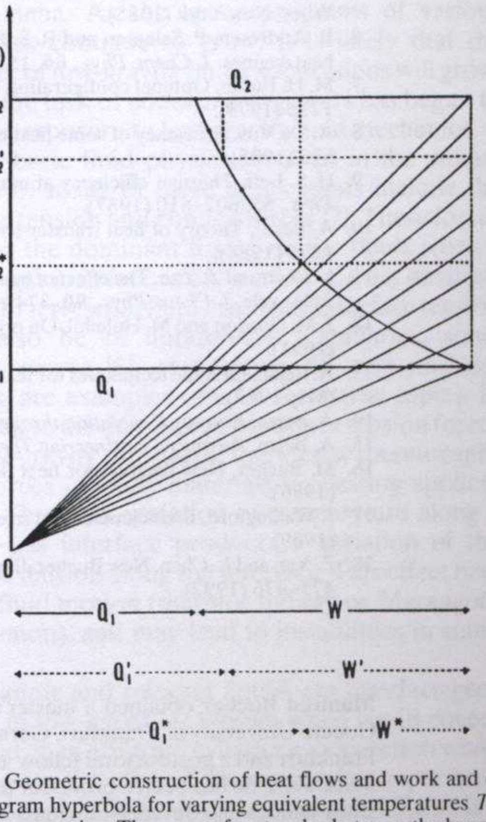

Fig. 2. Geometric construction of heat flows and work and of the diagram hyperbola for varying equivalent temperatures T_2' of a Carnot engine. The areas of rectangles between the hyperbola and the top right corner represent the corresponding power output. The reversible case (solid lines), a general irreversible case (dashed lines), and the case with maximum power (dotted lines) are explictly shown.

Yan and Chen observe that, since $p \to 0$ for $T_2' \to T_2$ as well as for $T_2' \to T_1$, there must be a maximum power p^*. The authors show that there are always pairs of equivalent temperatures, say T_{22}' and T_{21}', that lead to the same power output p'. Expressing these temperatures in terms of system properties and equating $T_{22}' = T_{21}'$ gives the formulas for p^* and for η^*, Eq. (6).

The present diagram permits a purely graphical derivation of η^* and related quantities for maximum-power output of a Carnot engine. This is illustrated in Fig. 2. Ten values of T_2', ranging equidistantly from T_2 to T_1, are chosen. The intersections of the slanted lines with the bottom bar and the corresponding points P straight above are marked with dots. Connecting the points P by a smooth curve gives the hyperbola which determines the rectangles A to be maximized in area. One such rectangle A—always spanned by a point P on the hyperbola and the top right corner of the diagram—is explicitly shown by dashed lines for one general value of T_2'.

Because of the symmetric diagram construction in Fig. 2 ($g = 1$), the rectangle A becomes a square when the associated rectangle $_O A^P$ is a square and as their diagonals become collinear. Thus, with $g = 1$, the rectangle of maximal area is a square, denoted as A* and shown in Fig. 2 by dotted lines. Also, the slanted line for the reversible Carnot cycle ($T_2' = T_2$) is then at 45°, serving as diagonals of the $_O A^P$ and A* squares and intersecting the hyperbola at point P for the A* square.

While the power output is proportionally represented in the present diagram by the area of a rectangle, the corresponding efficiency is concurrently shown by the length ratio of lines. The three bars below the diagram in Fig. 2 show the projected division of heat outflow and work output for the reversible case, a general irreversible case, and the case with maximum power. The portion of a work segment relative to the length of the total bar illustrates the respective efficiency: η of Eq. (1), η' of Eq. (10), and η^* of Eq. (6).

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