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Inverse Kinematics of a Robot Manipulator on a CAD System*

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This paper presents a computer-aided graphical approach to the inverse kinematics of serial manipulators, with a CCC manipulator as an example. The approach provides visualization of the mechanism under study and leads to solutions that are as accurate as those obtained by analytical methods.

EDUCATIONAL SUMMARY

1. The paper discusses materials for a course in computer-aided design of spatial mechanisms and robots.
2. Students of the mechanical engineering and industrial engineering departments are taught in this course.
3. Level of the course (year)
   4th year or 5th year
4. The material is presented with transparencies or computers.
5. The material is presented in a regular course.
6. At least 2hr is required to cover the material.
7. Students should do at least 6 hr homework or revision for the material.
8. The paper describes a new approach to inverse kinematics of serial robot manipulators.
9. The standard text recommended in the course, in addition to the author's notes is G. N. Sandor and A. G. Erdman, *Advanced Mechanism Design: Analysis and Synthesis*, Prentice Hall.
10. The material is partly covered in the text.

INTRODUCTION

INVERSE kinematics of a serial manipulator is a mathematical transformation of a system from world coordinates to joint coordinates. The world coordinates are the coordinates of the position and the orientations of the end of the manipulator. The joint coordinates are the magnitudes of the joint variables, which are directly controlled by the joint actuators of a robot manipulator [1]. In robot applications, the world coordinates, often defined by the user based on the requirements, need to be converted to the joint coordinates for robot controls.

Extensive analytical study has been made on the inverse kinematics of a number of serial robot manipulators [2, and refs therein]. Two categories of approaches have been used to study mechanisms: analytical and graphical. Graphical approaches are well known for many two-dimensional problems [3]. These approaches have a major advantage over analytical approaches in that they allow visualization during the design process, which means much better understanding of the problem under study and easier detection of errors. Instead of dealing with matrices and their equations extracted from the physical problem, graphical approaches deal directly with the geometry of the mechanism. Recently, attempts have been made to apply graphical approaches to the study of three-dimensional mechanisms on three-dimensional computer-aided design (CAD) systems [4, 5]. The accuracy of graphical work performed on the CAD system is supported by the system and visualization is much enhanced on the CAD system as the user is able to define and select various views.

This paper addresses computer-aided graphical approaches to inverse kinematics of serial manipulators, with a cylindrical joint–cylindrical joint–cylindrical joint (CCC) serial manipulator as an example. Before presenting the graphical procedure, an analytical procedure for inverse kinematics of the CCC manipulator will be discussed. The analytical procedure presented in this paper is an improvement of the one suggested in [2]. The analytical results of the example problem will be compared with the results obtained by the graphical method proposed.

CCC SERIAL MANIPULATOR

A kinematic schematic of the CCC manipulator is shown in Fig. 1, in which

S_i and θ_i ($i = 1, 2, 3$) the linear displacement vector and the angular displacement of the cylindrical joint *i*. These are the six joint variables of a CCC mechanism.

a_i ($i = 1, 2$) the vector perpendicular to the vector S_i and the vector S_{i+1} and joining the two vectors. The magnitudes of the vectors a_1 and a_2 are constants of a CCC mechanism.

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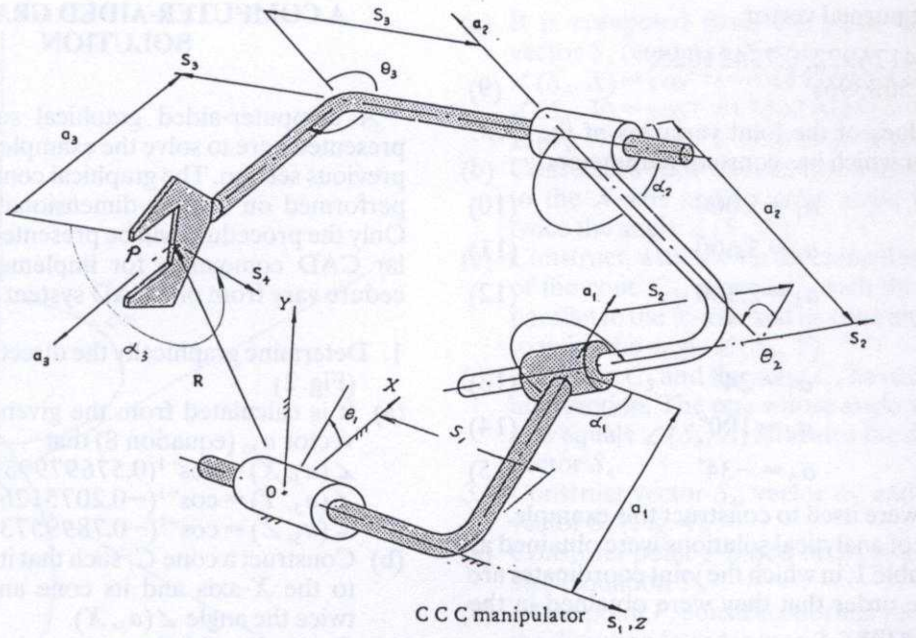

Fig. 1. The CCC manipulator.

α_i — $(i = 1, 2)$ the angle from the vector S_i to the vector S_{i+1}. α_1 and α_2 are constants of a CCC mechanism.

a_3 — the approach vector of the end effector. Its magnitude is a constant of a CCC mechanism. Its direction in space in one world coordinate of the manipulator.

S_4 — the normal vector of the end-effector. Its direction in space is another world coordinate of the manipulator. Since its magnitude has no significance, it is normally a unit vector.

α_3 — the angle from S_3, the axis of the cylindrical joint 3, to S_4, the normal to the end-effector. It is a constant of a CCC mechanism.

R — the position vector of the end point of the manipulator. It is yet another world coordinate of a CCC mechanism.

The reference Cartesian coordinate system is set at the joint 1 with the Z-axis coincident with the axis of the joint.

In an inverse kinematic problem of a CCC manipulator, the position vector R of the end point of the manipulator

$$R = (P_x, P_y, P_z)^{\mathrm{T}} \tag{1}$$

the unit approach vector a_{30}

$$a_{30} = (L_x, L_y, L_z)^{\mathrm{T}} \tag{2}$$

and the unit normal vector S_4

$$S_4 = (N_x, N_y, N_z)^{\mathrm{T}} \tag{3}$$

are given to find the angular joint variables θ_1, θ_2, θ_3 and the linear variables S_1, S_2, S_3 of the three cylindrical joints. The fact that a_{30} is a unit vector, S_4 is a unit vector, and vectors a_3 and S_4 are perpendicular to each other implies three constraint equations:

$$L_x^2 + L_y^2 + L_z^2 = 1 \tag{4}$$

$$N_x^2 + N_y^2 + N_z^2 = 1 \tag{5}$$

$$L_xN_x + L_yN_y + L_zN_z = 0 \tag{6}$$

The number of independent variables in equations (1)–(3) is six, which matches the number of the unknown variables to be found.

AN ANALYTICAL SOLUTION

The analytical kinematic equations of the CCC manipulator and a procedure of performing inverse kinematics analytically are shown in the Appendix. Please note that the procedure suggested in the current paper is simpler than the one in [2].

The example below shows the analytical solutions of an inverse kinematics problem of a CCC manipulator. The solutions will be compared with the results of the graphical approach to be presented in this paper.

Example

Given the location vector of the end point

$$R = (3.83664944, -3.63552226, 2.29342276)^{\mathrm{T}} \tag{7}$$

the unit approach vector

$$a_{30} = (0.57697995, -0.20751260, -0.78995738)^{\mathrm{T}} \tag{8}$$

and the unit normal vector

$$S_4 = (-0.44176922, 0.73421025, -0.51553396)^T \quad (9)$$

find the values of the joint variables of the CCC manipulator which has constant parameters

$$a_1 = 3.000 \quad (10)$$

$$a_2 = 3.600 \quad (11)$$

$$a_3 = 2.300 \quad (12)$$

and

$$\alpha_1 = 24° \quad (13)$$

$$\alpha_2 = 100° \quad (14)$$

$$\alpha_3 = -34° \quad (15)$$

Data in [2] were used to construct this example.

Two sets of analytical solutions were obtained as shown in Table 1, in which the joint coordinates are listed in the order that they were obtained in the solution process.

Table 1. The analytical solutions of the inverse kinematics example problem

θ_2	78.021087°	−78.021087°
θ_1	95.81429396°	262.07073621°
θ_3	210.29178512°	258.7070207°
S_2	15.37792766	−15.37792766
S_3	−7.41508406	1.00205055
S_1	−13.16323637	19.83339477

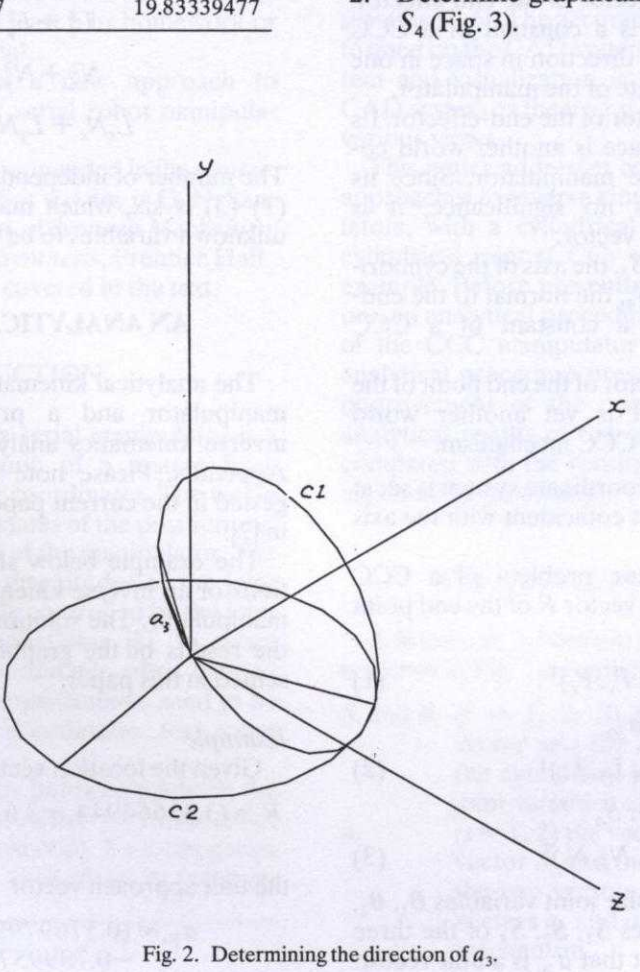

Fig. 2. Determining the direction of a_3.

A COMPUTER-AIDED GRAPHICAL SOLUTION

A computer-aided graphical solution will be presented here to solve the example problem in the previous section. The graphical constructions were performed on a three-dimensional CAD system. Only the procedure will be presented here. Particular CAD commands for implementing the procedure vary from one CAD system to another.

1. Determine graphically the direction of vector a_3 (Fig. 2)

(a) It is calculated from the given unit approach vector a_{30} (equation 8) that
$\angle(a_3, X) = \cos^{-1}(0.57697995) = 54.7616°$
$\angle(a_3, Y) = \cos^{-1}(-0.20751260) = 101.977°$
$\angle(a_3, Z) = \cos^{-1}(-0.78995738) = 142.181°$

(b) Construct a cone C_1 such that its axis is parallel to the X-axis and its cone angle is equal to twice the angle $\angle(a_3, X)$.

(c) Construct, with the vertex coincident with that of the cone C_1, a cone C_2 such that its axis is parallel to the Y-axis and its cone angle is equal to twice the angle $\angle(a_3, Y)$.

(d) The cone C_1 and the cone C_2 have two lines of intersection. The one whose angle with the Z-axis equals $\angle(a_3, Z)$ indicates the direction of vector a_3.

2. Determine graphically the direction of vector S_4 (Fig. 3).

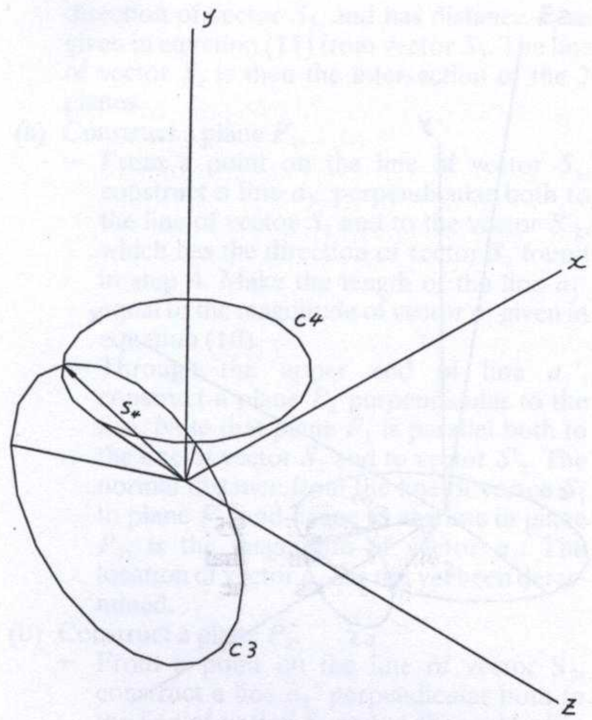

Fig. 3. Determining the direction of S_4.

(a) It is computed from the given unit normal vector S_4 (equation 9) that
$\angle(S_4, X) = \cos^{-1}(-0.44176922) = 116.217°$
$\angle(S_4, Y) = \cos^{-1}(0.73421025) = 47.7594°$
$\angle(S_4, Z) = \cos^{-1}(-0.51553396) = 121.033°$

(b) Construct a cone C_3 such that its axis is parallel to the X-axis and its cone angle is equal to twice the angle $\angle(S_4, X)$.

(c) Construct, with the vertex coincident with that of the cone C_3, a cone C_4 such that its axis is parallel to the Y-axis and its cone angle is equal to twice the angle $\angle(S_4, Y)$.

(d) The cone C_3 and the cone C_4 have two lines of intersection. The one whose angle with the Z-axis equals $\angle(S_4, Z)$ indicates the direction of vector S_4.

3. Construct vector S_4, vector a_3, and the line of vector S_3 (Fig. 4).

(a) Construct point P based on its given coordinates (equation 7).

(b) At the point P, construct normal vector S_4 with the direction found in step 2. The length can be arbitrary.

(c) From the point P, construct vector a_3 with the direction found in step 1 and the length given in equation (12).

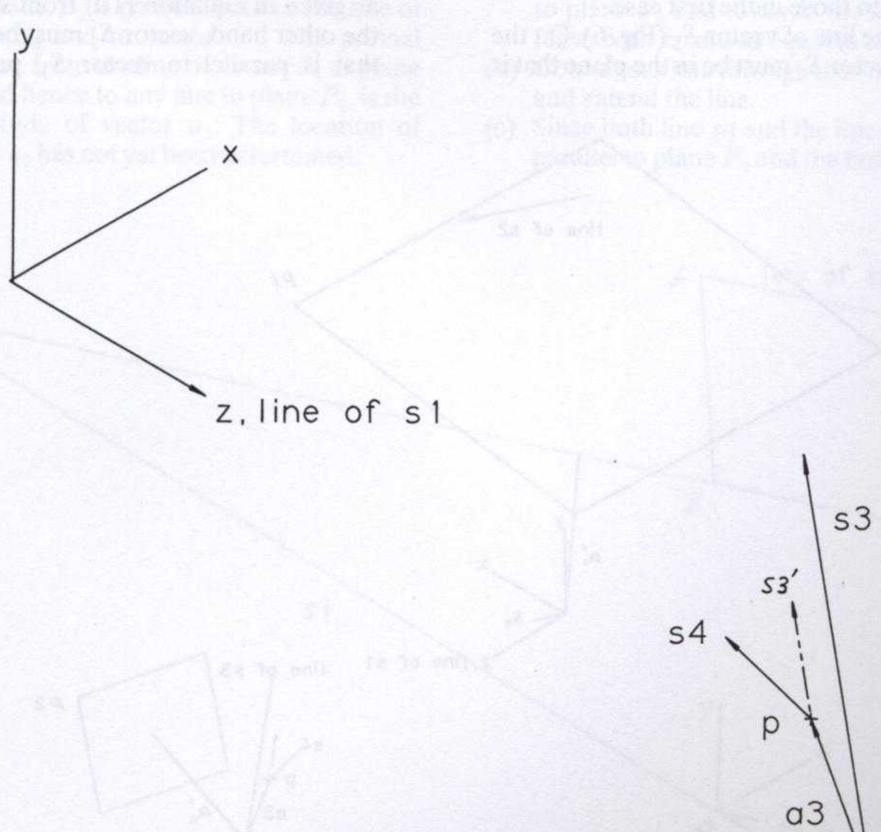

Fig. 4. Locating a_3 and determining the line of S_3.

(d) Rotate the vector S_4 about the vector a_3 by an angle equal to the negative of α_3, which is the angle from the vector S_3 to the vector S_4 and is given in equation (15). The resulting vector S_3' indicates the direction of vector S_3.

(e) Construct, at the other end of vector a_3, a line parallel to vector S_3'. This is the line of vector S_3, whose length is to be determined. In Fig. 4, as in Fig. 1, the Z-axis coincides with the line of vector S_1. The length of the vector is to be determined. To determine the lengths of vector S_1 and vector S_3, the line of vector S_2 needs to be found.

4. Determine graphically the direction of vector S_2, which forms angle α_1 given in equation (13) with vector S_1 an angle α_2 given in equation (14) with vector S_3 (Fig. 5).

(a) Construct a cone C_5 such that its axis is parallel to the line of vector S_1 (i.e. the Z-axis) and its cone angle is equal to $2\alpha_1$.

(b) Construct, with the vertex coincident with that of cone C_5, a cone C_6 such that its axis is parallel to the line of vector S_3 found in step 3 and whose cone angle is equal to $2\alpha_2$.

(c) The cone C_5 and the cone C_6 have two lines of intersection $S_2^{(1)}$ and $S_2^{(2)}$. They are the two valid solutions for the direction of vector S_2. We will show, step by step, the rest of the work with $S_2^{(1)}$ being the direction of vector S_2 and show only the final result with $S_2^{(2)}$ being the direction of S_2. The constructions in the second case are very similar to those in the first case.

5. Construct the line of vector S_2 (Fig. 6). On the one hand, vector S_2 must be in the plane that is

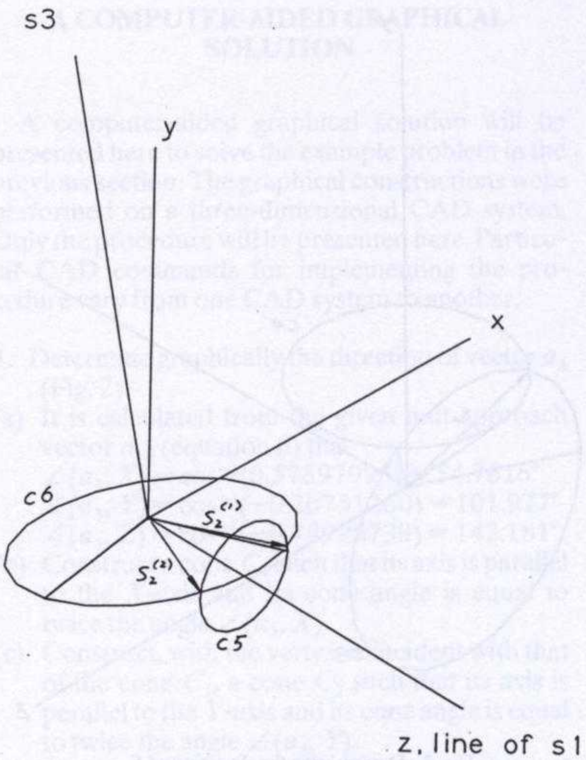

Fig. 5. Determining the direction of S_2.

parallel to vector S_1, parallel to the direction of vector S_2 found in step 4, and has distance a_1 as given in equation (10) from vector S_1. On the other hand, vector S_2 must be in the plane that is parallel to vector S_3, parallel to the

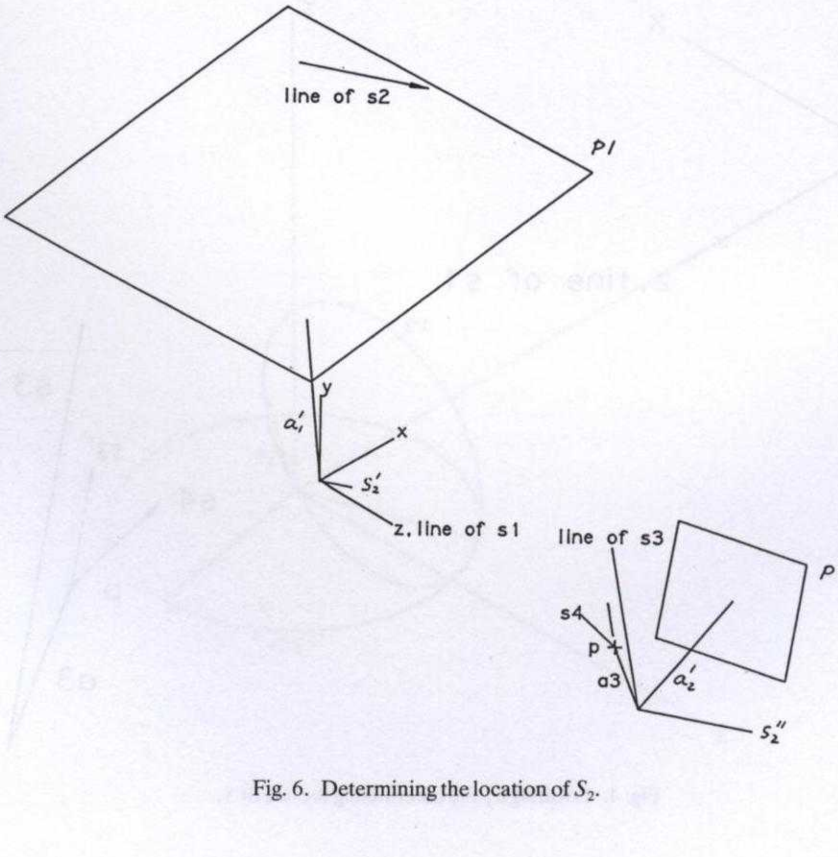

Fig. 6. Determining the location of S_2.

direction of vector S_2, and has distance a_2 as given in equation (11) from vector S_3. The line of vector S_2 is then the intersection of the 2 planes.

(a) Construct a plane P_1.

— From a point on the line of vector S_1, construct a line a_1' perpendicular both to the line of vector S_1 and to the vector S'_2, which has the direction of vector S_2 found in step 4. Make the length of the line a_1' equal to the magnitude of vector a_1 given in equation (10).

— Through the upper end of line a_1', construct a plane P_1 perpendicular to the line. Note that plane P_1 is parallel both to the line of vector S_1 and to vector S'_2. The normal distance from the line of vector S_1 to plane P_1, and hence to any line in plane P_1, is the magnitude of vector a_1. The location of vector a_1 has not yet been determined.

(b) Construct a plane P_2.

— From a point on the line of vector S_3, construct a line a_2' perpendicular both to the line of vector S_3 and to the vector S''_2, which has the direction of vector S_2 found in step 4. Make the length of the line a_2' equal to the magnitude of vector a_2 given in equation (11).

— Through the upper end of line a_2', construct a plane P_2 perpendicular to the line. Note that plane P_2 is parallel both to the line of vector S_3 and to vector S''_2. The normal distance from the line of vector S_3 to plane P_2, and hence to any line in plane P_2, is the magnitude of vector a_2. The location of vector a_2 has not yet been determined.

(c) Construct the intersetion of planes P_1 and P_2. The line of intersection of the two planes is the line of vector S_2 since (i) it has the direction of vector S_2 because both planes P_1 and P_2 are parallel to the direction of the vector; (ii) its distance to the line of vector S_1 is a_1; and (iii) its distance to the line of vector S_3 is a_2. Incidentally, on the CAD system the two planes need not be shown to meet each other in order to determine their intersection.

6. Construct the point of connection of vector a_1 and vector S_1 and determine the length of vector S_1 (Fig. 7).

(a) From two points *f* and *K* on the line of vector S_2, construct two lines *fg* and *kl* perpendicular to plane P_1 and towards the line of vector S_1. Have both *fg* and *kl* equal to a_1.

(b) Construct a line through the end points *l* and *g* and extend the line.

(c) Since both line *lg* and the line of vector S_1 are parallel to plane P_1 and the both have the same distance a_1 to the plane, the two lines meet at point *m*.

(d) Point *m* is the point of connection of vector a_1 and vector S_1. Line *om* is the length of vector S_1.

7. Construct the point of connection of vector a_2 and vector S_3 and determine the length of vector S_3 (Fig. 8).

(a) From two points *r* and *t* on the line of vector S_2, construct two lines *rs* and *tu* perpendicular to plane P_2 and towards the line of vector S_3. Have both *rs* and *tu* equal to a_2.

(b) Construct a line through the end points *s* and *u* and extend the line.

(c) Since both line *su* and the line of vector S_3 are parallel to plane P_2 and the both have the same

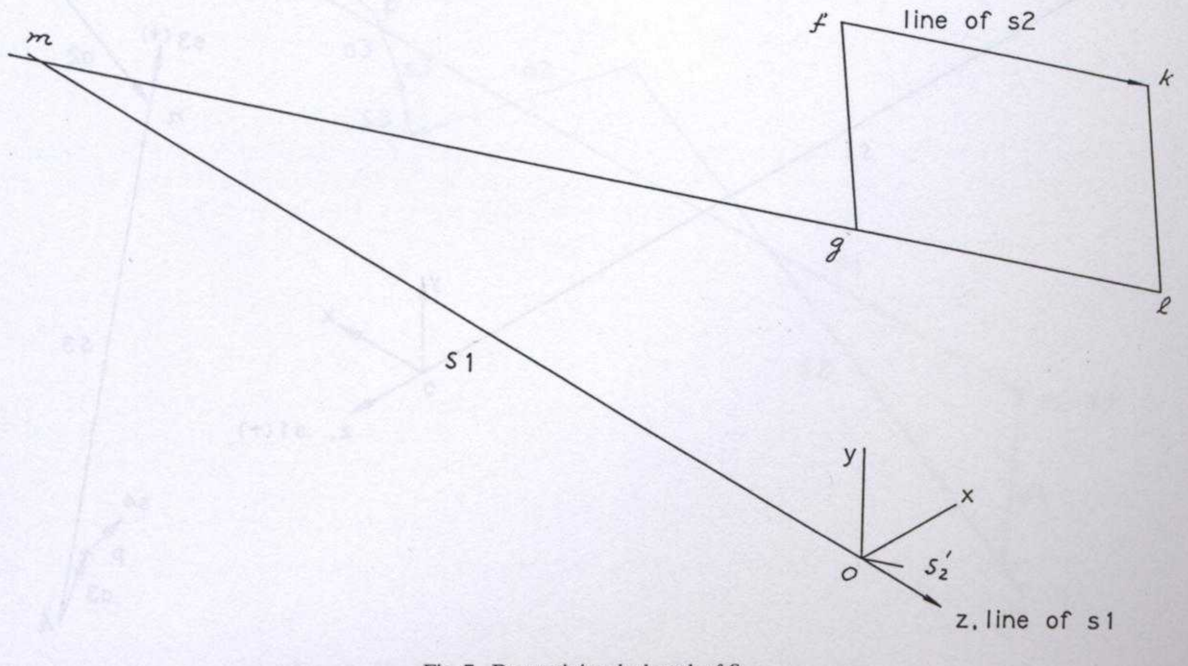

Fig. 7. Determining the length of S_1.

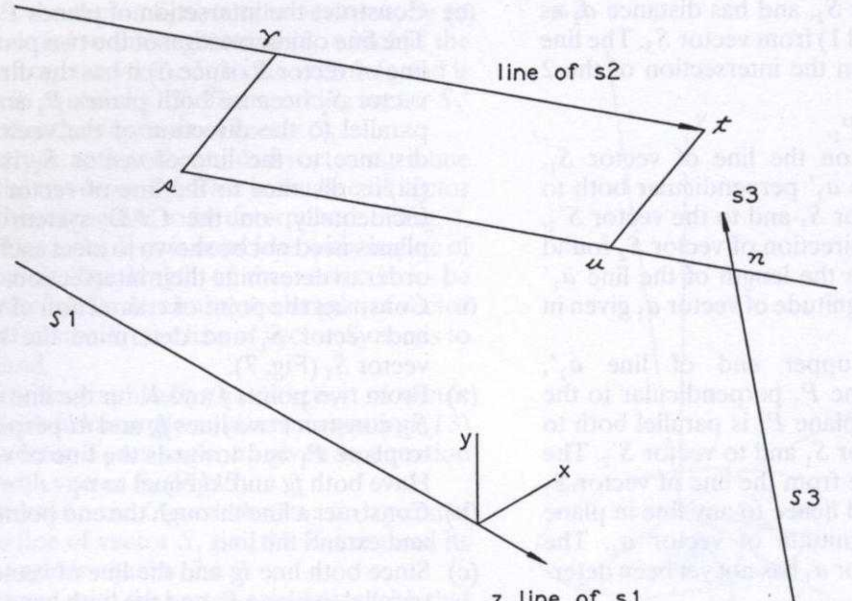

Fig. 8. Determining the length of S_3.

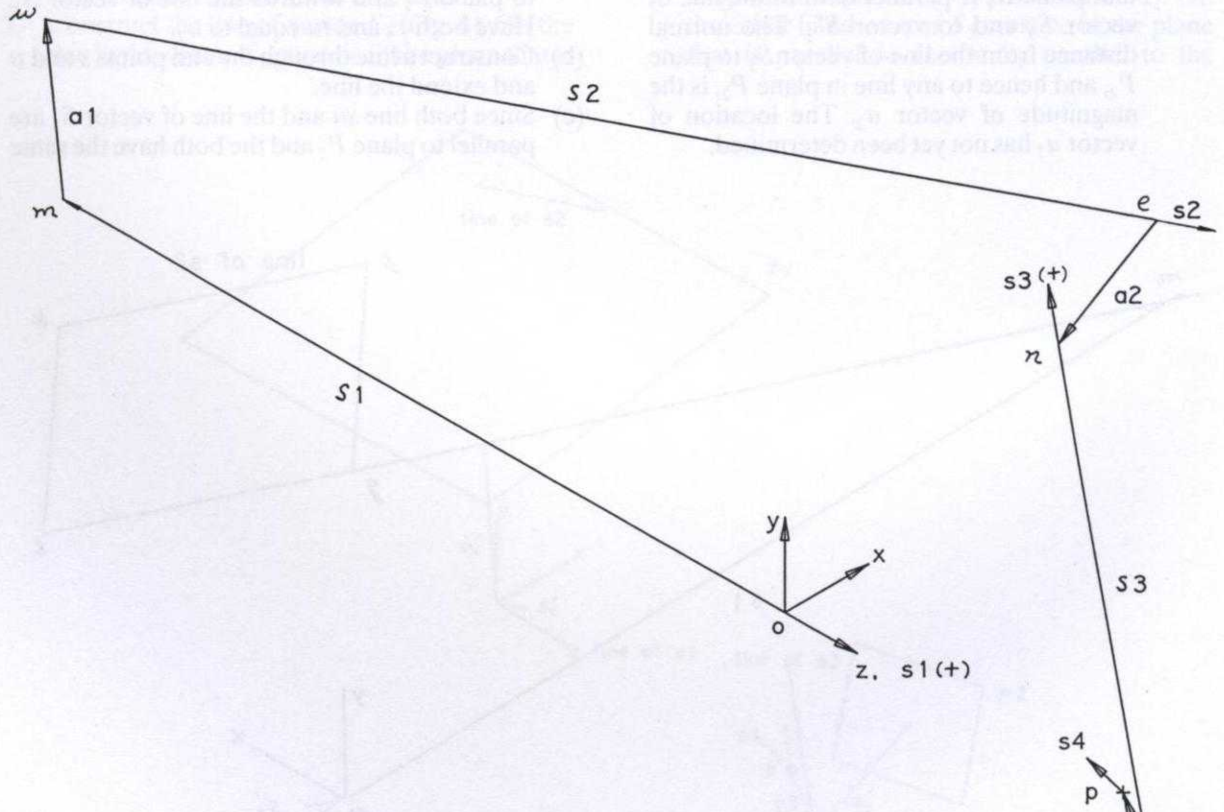

Fig. 9. Determining the length of S_2.

distance a_2 to the plane, the two lines meet at a point n.

(d) Point n is the point of connection of vector a_2 and vector S_3. Line hn is the length of vector S_3.

8. Construct vector a_1 and vector a_2 (Fig. 9).

(a) From point m on vector S_1, the point of connection of vector S_1 and vector a_1 found in step 6, construct a line perpendicular to the vector S_1 and to the line of vector S_2. The line intersects with the line of vector S_2 at a point w. Point w is an end point of vector S_2. Vector a_1 is also located.

(b) From point n on vector S_3, the point of connection of vector S_3 and vector a_2 found in step 7, construct a line perpendicular to S_3 and S_2. The line intersects with the line of vector S_2 at a point e. Point e is the other end point of vector S_2. Vector S_2 is determined. Vector a_2 is located.

The constructions are now complete. The kinematic skeleton of the mechanism configuration is shown in Fig. 9. The values of the joint variables found on the CAD system are listed below in the order of the variables in Table 1:

$\theta_2 = 78.021°$
$\theta_1 = 95.8142°$
$\theta_3 = 210.2918°$
$S_2 = 15.3779$
$S_3 = 7.41506$ (negative direction)
$S_1 = 13.1632$ (negative direction)

which are shown to be accurate as the analytical results in Table 1, except that fewer significant digits were used in the CAD constructions.

The second solution of the inverse kinematics problem, where line $S_2^{(2)}$ instead of line $S_2^{(1)}$ is chosen from Fig. 5 for the direction of vector S_2, is shown in Fig. 10.

CONCLUSIONS

A complete procedure of performing inverse kinematics for a CCC robot manipulator on the CAD system has been shown. The procedure can be written into a script file, whose commands are CAD system-dependent, for automatic execution. The methods presented were based on basic concepts in spatial geometry. The joint coordinates obtained on the CAD system, a powerful tool for spatial construction and visualization, were as accurate as those obtained by analytical methods up to the number of significant figures used in the constructions. The high accuracy is due to the fact that measurements on the CAD system are the result of digital computations. Graphical methods have been generally criticized for their low accuracy. This problem should no longer exist with use

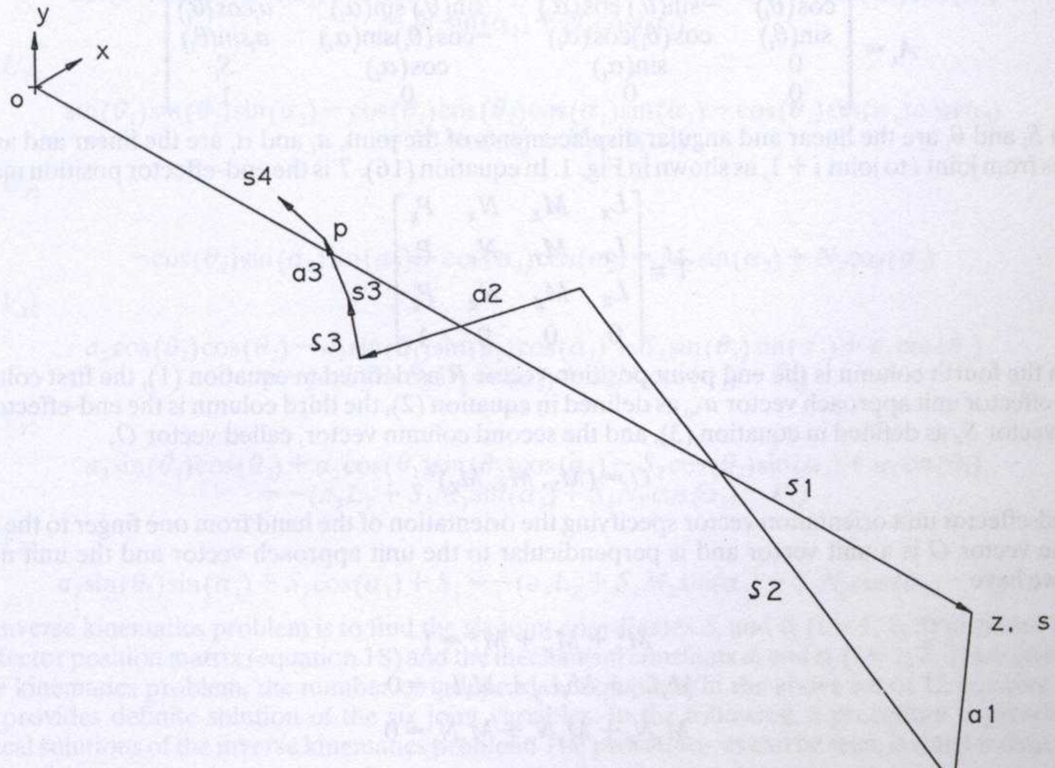

Fig. 10. The mechanism configuration with a different line of S_2.

of the CAD system. For various kinematics problems of spatial mechanisms, graphical methods on the CAD system appear to be very promising supplements to analytical methods.

It should be noted that not every three-dimensional mechanism kinematics problem can be solved on the CAD system in a straightforward manner. Problems that cannot be solved analytically without iterations would not be solved easily on the CAD system.

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APPENDIX

Analytical approaches for inverse kinematics of the CCC robot manipulator are based on the following well-known homogeneous transformation equation [1, 2]:

$$A_1 A_2 A_3 = T \tag{16}$$

In the above equation A_i ($i = 1, 2, 3$) is the homogeneous transformation matrix of joint i:

$$A_i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & S_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

in which S_i and θ_i are the linear and angular displacements of the joint, a_i and α_i are the linear and angular distances from joint i to joint $i+1$, as shown in Fig. 1. In equation (16), T is the end-effector position matrix

$$T = \begin{bmatrix} L_X & M_X & N_X & P_X \\ L_Y & M_Y & N_Y & P_Y \\ L_Z & M_Z & N_Z & P_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

in which the fourth column is the end point position vector R as defined in equation (1), the first column is the end-effector unit approach vector a_{30} as defined in equation (2), the third column is the end-effector unit normal vector S_4 as defined in equation (3), and the second column vector, called vector O,

$$O = (M_X, M_Y, M_Z)^T \tag{19}$$

is the end-effector unit orientation vector specifying the orientation of the hand from one finger to the other. Since the vector O is a unit vector and is perpendicular to the unit approach vector and the unit normal vector, we have

$$M_x^2 + M_y^2 + M_z^2 = 1 \tag{20}$$

$$M_x L_x + M_y L_y + M_z L_z = 0 \tag{21}$$

$$M_x N_x + M_y N_y + M_z N_z = 0 \tag{22}$$

With the above three equations, the unit orientation vector O can be determined once the unit approach vector a_{30} and the unit normal vector S_4 are known.

For inverse kinematics, equation (16) is rearranged to become

$$A_1 A_2 = T A_3^{-1} \tag{23}$$

After the two 4 × 4 matrices on each side are multiplied by each other, the above equation has the form

$$\begin{bmatrix} J_{X1} & K_{X1} & U_{X1} & V_{X1} \\ J_{Y1} & K_{Y1} & U_{Y1} & V_{Y1} \\ J_{Z1} & K_{Z1} & U_{Z1} & V_{Z1} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} J_{X2} & K_{X2} & U_{X2} & V_{X2} \\ J_{Y2} & K_{Y2} & U_{Y2} & V_{Y2} \\ J_{Z2} & K_{Z2} & U_{Z2} & V_{Z2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (24)$$

This matrix equation is equivalent to 12 non-trivial scalar equations, which will be listed below.

$J_{x1} = J_{x2}$

or its expansion

$$\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\cos(\alpha_1) = \cos(\theta_3)L_X + \sin(\theta_3)(N_X\sin(\alpha_3) - M_X\cos(\alpha_3)) \quad (25)$$

$J_{Y1} = J_{Y2}$

$$\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\cos(\alpha_1)\sin(\theta_2) = \cos(\theta_3)L_Y + \sin(\theta_3)(N_Y\sin(\alpha_3) - M_Y\cos(\alpha_3)) \quad (26)$$

$J_{Z1} = J_{Z2}$

$$\sin(\alpha_1)\sin(\theta_2) = \cos(\theta_3)L_Z + \sin(\theta_3)(N_Z\sin(\alpha_3) - M_Z\cos(\alpha_3)) \quad (27)$$

$K_{X1} = K_{X2}$

$$-\cos(\theta_1)\sin(\theta_2)\cos(\alpha_2) - \sin(\theta_1)\cos(\theta_2)\cos(\alpha_1)\cos(\alpha_2) + \sin(\theta_1)\sin(\alpha_1)\sin(\alpha_2) = \sin(\theta_3)L_X + \cos(\theta_3)(M_X\cos(\alpha_3) - N_X\sin(\alpha_3)) \quad (28)$$

$K_{Y1} = K_{Y2}$

$$-\sin(\theta_1)\sin(\theta_2)\cos(\alpha_2) + \cos(\theta_1)\cos(\theta_2)\cos(\alpha_1)\cos(\alpha_2) - \cos(\theta_1)\sin(\alpha_1)\sin(\alpha_2) = \sin(\theta_3)L_Y + \cos(\theta_3)(M_Y\cos(\alpha_3) - N_Y\sin(\alpha_3)) \quad (29)$$

$K_{Z1} = K_{Z2}$

$$\cos(\theta_2)\sin(\alpha_1)\cos(\alpha_2) + \cos(\alpha_1)\sin(\alpha_2) = \sin(\theta_3)L_Z + \cos(\theta_3)(M_Z cos(\alpha_3) - N_Z\sin(\alpha_3)) \quad (30)$$

$U_{X1} = U_{X2}$

$$\cos(\theta_1)\sin(\theta_2)\sin(\alpha_2) + \sin(\theta_1)\cos(\theta_2)\cos(\alpha_1)\sin(\alpha_2) + \sin(\theta_1)\sin(\alpha_1)\cos(\alpha_2) = M_x\sin(\alpha_3) + N_X\cos(\alpha_3) \quad (31)$$

$U_{Y1} = U_{Y2}$

$$\sin(\theta_1)\sin(\theta_2)\sin(\alpha_2) - \cos(\theta_1)\cos(\theta_2)\cos(\alpha_1)\sin(\alpha_2) - \cos(\theta_1)\sin(\alpha_1)\cos(\alpha_2) = M_Y\sin(\alpha_3) + N_Y\cos(\alpha_3) \quad (32)$$

$U_{Z1} = U_{Z2}$

$$-\cos(\theta_2)\sin(\alpha_1)\sin(\alpha_2) + \cos(\alpha_1)\cos(\alpha_2) = M_Z\sin(\alpha_3) + N_Z\cos(\alpha_3) \quad (33)$$

$V_{X1} = V_{X2}$

$$a_2\cos(\theta_1)\cos(\theta_2) - a_2\sin(\theta_1)\sin(\theta_2)\cos(\alpha_1) + S_2\sin(\theta_1)\sin(\alpha_2) + a_1\cos(\theta_1) = -(a_3L_X + S_3M_X\sin(\alpha_3) + S_3N_X\cos(\alpha_3) - P_X) \quad (34)$$

$V_{Y1} = V_{Y2}$

$$a_2\sin(\theta_1)\cos(\theta_2) + a_2\cos(\theta_1)\sin(\theta_2)\cos(\alpha_1) - S_2\cos(\theta_1)\sin(\alpha_1) + a_1\sin(\theta_1) = -(a_3L_Y + S_3M_Y\sin(\alpha_3) + S_3N_Y cos(\alpha_3) - P_Y) \quad (35)$$

$V_{Z1} = V_{Z2}$

$$a_2\sin(\theta_2)\sin(\alpha_1) + S_2\cos(\alpha_1) + S_1 = -(a_3L_Z + S_3M_Z\sin(\alpha_3) + S_3N_Z\cos(\alpha_3) - P_Z) \quad (36)$$

An inverse kinematics problem is to find the six joint coordinates S_i and θ_i ($i = 1, 2, 3$) provided that the end-effector position matrix (equation 18) and the mechanism constants a_i and α_i ($i = 1, 2, 3$) are given. In an inverse kinematics problem, the number of independent equations in the above set of 12 equations is six, which provides definite solution of the six joint variables. In the following, a procedure is presented for analytical solutions of the inverse kinematics problem. The procedure, as can be seen, is simpler than the one in [2].

Procedure

1. Solve equation (33) for the value of $\cos(\theta_2)$, from which two solutions of angle θ_2 can be obtained. Equation (33) has been selected to be used first since it contains only one joint variable θ_2. In fact, U_{Z1}, the unexpanded form of the left-hand side of equation (33), is the scalar product of the third row vector of the matrix A_1 and the third column vector of the matrix A_2, as can be seen from equations (24), (23) and (17).

For each of the two solutions of θ_2, do the following:

2. Solve equations (31) and (32) simultaneously for the values of $\sin(\theta_1)$ and $\cos(\theta_1)$, and determine the value of θ_1.
3. Solve equations (27) and (30) simultaneously for the values of $\sin(\theta_3)$ and $\cos(\theta_3)$, and determine the value of θ_3
4. Solve equations (34) and (35) simultaneously for the values of S_2 and S_3.
5. Solve equations (36) for the value of S_1.