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Magnetostatic Torque Experienced by an Electric Circuit*

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A simple derivation is given for the torque experienced by an electric current loop of arbitrary shape, not necessarily confined to a plane, when it is placed in a magnetostatic field, which is not necessarily uniform, in terms of the magnetic moment of the loop. The result is illustrated with the help of an example.

1. The paper discusses materials for a course in:
   Electromagnetic fields and/or electromechanical dynamics
2. Students of the following departments are taught in this course:
   Electrical engineering
3. Level of the course (year)
   Third-year undergraduate
4. Mode of presentation:
   Class lecture
5. Is the material presented in a regular or elective course:
   Regular course
6. Class or hours required to cover the material:
   The material forms a part of the discussion of forces and torques on electric circuits when they are subjected to magnetic fields. Total time devoted to the topic is three or four lectures.
7. Student homework or revision hours required for the materials:
   Approximately 4 hours
8. Description of the novel aspects presented in your paper:
   The torque acting on a current circuit placed in an arbitrary magnetic field is expressed in terms of the magnetic moment of the circuit. A simple derivation of the formula for torque is provided, and the result is illustrated by an example.
9. The standard text recommended in the course, in addition to author's notes:
   Texts vary. Current texts are: electromagnetic fields—C. R. Paul and S. A. Nasar, *Introduction to Electromagnetic Fields*, McGraw-Hill (1987); electromechanical dynamics—F. R. Bergseth and S. S. Venkata, *Introduction to Electric Energy Devices*, Prentice Hall (1987).
10. The material is/is not covered in the text.
    Not covered.

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INTRODUCTION

CALCULATION of torques on current-carrying circuits when they are placed in magnetic fields is of interest to electrical engineers from the viewpoints of both theory and practical applications. One method for evaluating the torque is in terms of the magnetic moment of the loop. The subject is commonly discussed in texts on electromagnetism, in varying degrees of depth. The following general observations can be made:

1. Several texts [1–6] assume the loop to lie in a plane. (As noted in [7], this restriction is not necessary.)
2. In most texts, including those cited above, the treatment is restricted to *uniform* magnetic fields.
3. Some texts do not clarify the fact that when an electric circuit is placed in a non-uniform magnetic field, there acts on the circuit, in general, a force of translation in addition to a couple, with the result that the torque is dependent on the choice of the reference point. Only when the magnetic field is uniform does the force of transmission become zero, in which case the torque reduces to a couple and the reference point need not be specified (see, e.g. [8]).

The following formula (without proof) was given in [9] for torque on a current loop, whose shape can be chosen at will, when it is placed in a non-uniform magnetic field.

$$\mathbf{T} = \int_s d\mathbf{M} \times \mathbf{B} + \int_s \mathbf{r} \times (d\mathbf{M} \cdot \nabla)\mathbf{B} \qquad (1)$$

In (1), $d\mathbf{M}$ is the magnetic moment of a differential current loop enclosing an area $d\mathbf{a}$, i.e. $d\mathbf{M} = I\,d\mathbf{a}$, and the integrals are over a surface S spanning the loop. This result assumes that $\nabla \times \mathbf{B} = 0$ over S, which is the case if there is no current density over S. The first integral in (1) represents the couple acting on the loop, and has no reference to the origin of coordinates. The second integral may be interpreted as the contribution to the torque arising from forces of translation, $d\mathbf{F}$, acting on differential current loops which together constitute the current loop: $d\mathbf{F} = (d\mathbf{M} \cdot \nabla)\, \mathbf{B}$. This part of the total torque depends on the choice of origin. The second integral vanishes if **B** is uniform, but must be taken into account in computing the total torque if **B** is not uniform.

This paper presents a derivation of the formula and an example to illustrate the result.

DERIVATION OF THE FORMULA FOR TORQUE

In this section, a simple proof of (1), suitable for undergraduate classroom presentation, is offered, based on consideration of forces on the four sides of a differential current loop.

Let a rectangular loop with sides dx and dy and centered at the point $P(x_0, y_0, z_0)$ lie in a plane parallel to the xy plane of a Cartesian coordinate system in the presence of a non-uniform magnetic field of flux density **B** (Fig. 1). The Lorentz force on the side 12 is:

$$d\mathbf{F}_{12} = I\, dy\, \mathbf{a}_y \times \mathbf{B}(x_0 + dx/2) = I\, dy[\mathbf{a}_x B_z(x_0 + dx/2) - \mathbf{a}_z B_x(x_0 + dx/2)] \quad (2)$$

Expanding $B_z(x_0 + dx/2)$ and $B_x(x_0 + dx/2)$ in a Taylor's series about the point P and retaining only the first-order differential terms, one obtains:

$$d\mathbf{F}_{12} = I\, dy[\mathbf{a}_x B_z(P) - \mathbf{a}_z B_x(P) + (\mathbf{a}_x \partial B_z/\partial x - \mathbf{a}_z \partial B_x/dx)dx/2] \quad (3)$$

The torque associated with $d\mathbf{F}_{12}$ evaluated about the origin O is

$$d\mathbf{T}_{12} = \mathbf{r}_{12} \times \mathbf{F}_{12} \quad (4)$$

where $\mathbf{r}_{12}$ is the position vector of the center-point of the segment 1–2 of the loop in Fig. 1. If **r** designates the position vector of P, then $\mathbf{r}_{12} = \mathbf{r} + \mathbf{a}_x(dx/2)$. Thus:

$$d\mathbf{T}_{12}/I = dy\,\mathbf{r} \times [\mathbf{a}_x B_z(P) - \mathbf{a}_z B_x(P)] + \mathbf{r} \times (\mathbf{a}_x \partial B_z/\partial x - \mathbf{a}_z \partial B_x/\partial x)da/2 + \mathbf{a}_y B_x(P)da/2 \quad (5)$$

where the third and higher powers of the differential segments have been neglected. Similarly, the torque $d\mathbf{T}_{34}$ about O associated with the Lorentz force on segment 3–4 of the loop is given by:

$$d\mathbf{T}_{34}/I = dy\,\mathbf{r} \times [-\mathbf{a}_x B_z(P) + \mathbf{a}_z B_x(P)] + \mathbf{r} \times (\mathbf{a}_x \partial B_z/\partial x - \mathbf{a}_z \partial B_x/\partial x)da/2 + \mathbf{a}_y B_x(P)da/2 \quad (6)$$

Summing (5) and (6), we obtain:

$$(d\mathbf{T}_{12} + d\mathbf{T}_{34})/I = da[\mathbf{r} \times (\mathbf{a}_x \partial B_z/\partial x - \mathbf{a}_z \partial B_x/\partial x) + \mathbf{a}_y B_x(P)] \quad (7)$$

Similarly, the contribution to torque arising from forces on sides 2–3 and 4–1 of the current loop is obtained as:

$$(d\mathbf{T}_{23} + d\mathbf{T}_{41})/I = da[\mathbf{r} \times (\mathbf{a}_y \partial B_z/\partial y - \mathbf{a}_z \partial B_y/\partial y) - \mathbf{a}_x B_y(P)] \quad (8)$$

Adding (7) and (8), and recognizing that $\nabla \cdot \mathbf{B} = 0$, the total torque $d\mathbf{T}$ acting on the loop is expressed as

$$d\mathbf{T}/I = d\mathbf{a}[\mathbf{r} \times (\mathbf{a}_x \partial B_z/\partial x + \mathbf{a}_y \partial B_z/\partial y + \mathbf{a}_z \partial B_z/\partial z) + \mathbf{a}_y B_x(P) - \mathbf{a}_x B_y(P)] \quad (9)$$

If $\nabla \times \mathbf{B} = \mathbf{0}$, then, $\partial B_z/\partial x = \partial B_x/\partial z$, $\partial B_z/\partial y = \partial B_y/\partial z$, so that $\mathbf{a}_x \partial B_z/\partial x + \mathbf{a}_y \partial B_z/\partial y + \mathbf{a}_z \partial B_z/\partial z = (\partial \mathbf{B}/\partial z) = (\mathbf{a}_z \cdot \nabla)\mathbf{B}$, and:

$$d\mathbf{T}/I = da(\mathbf{r} \times \nabla_z \mathbf{B} + \mathbf{a}_z \times \mathbf{B}) \quad (10)$$

It follows immediately that, for an arbitrarily oriented differential current loop:

$$d\mathbf{T} = \mathbf{r} \times (d\mathbf{M} \cdot \nabla)\mathbf{B} + d\mathbf{M} \times \mathbf{B} \quad (11)$$

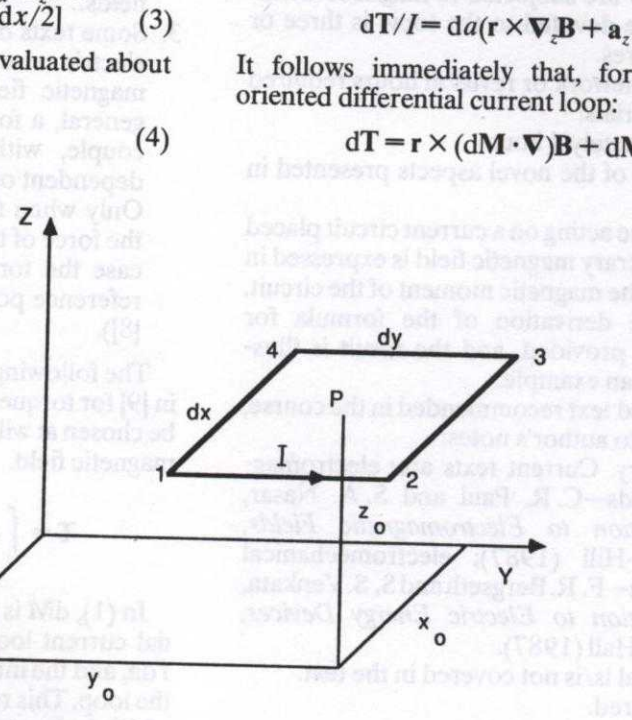

Fig. 1. A rectangular differential current loop placed in a non-uniform magnetic field.

The result (11) permits extension to a loop of finite dimensions, not necessarily planar. Inasmuch as a closed circuit can be considered to be made up of a large number of differential current loops, the torque on a closed circuit of arbitrary shape placed in a non-uniform field is given by (1).

AN EXAMPLE

Consider a single turn of an armature winding of a d.c. machine, with coil pitch p and axial length L, carrying a current I. Assume that the magnetic flux density is sinusoidally distributed on the armature surface, with a wavelength $2b$, as depicted in Fig. 2, in which a segment of the armature surface is approximated by a plane surface. On $y = 0$:

$$B_y = B_{y0}\sin\left(\frac{\pi x}{b} - \alpha\right) \qquad (12)$$

At the outset, one notes that, since $\nabla \times \mathbf{B} = 0$, a purely y-directed flux density of the form given by (1) cannot exist in the air gap. Assuming that there is no dependence of flux density on the z-coordinate, there must be an accompanying x-component of $\mathbf{B}$, such that:

$$\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = B_{y0}\frac{\pi}{b}\cos\left(\frac{\pi x}{b} - \alpha\right) \text{ on } y = 0 \qquad (13)$$

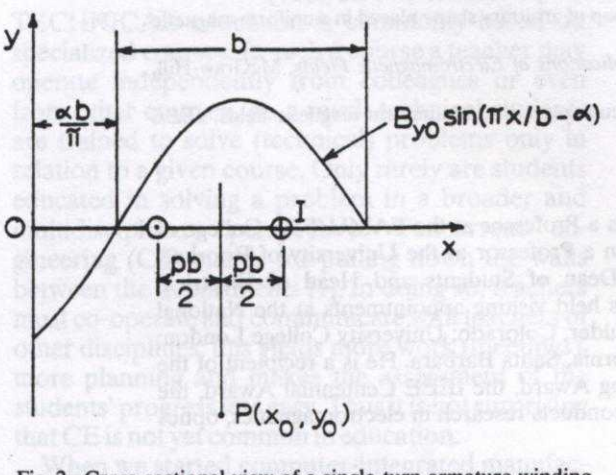

Fig. 2. Cross-sectional view of a single-turn armature winding of a d.c. machine in a sinusoidally distributed magnetic flux density.

A magnetic field which satisfies the conditions (12) and (13) can be generated by introducing a scalar magnetic potential u such that $\mathbf{B} = -\nabla u$. The potential u satisfies Laplace's equation in two dimensions, $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$, with the boundary condition $u = 0$ on $y = 0$. By the method of separation of variables, one readily finds that the above conditions are satisfied by

$$u = -B_{y0}\frac{b}{\pi}\sin\left(\frac{\pi x}{b} - \alpha\right)\sinh\left(\frac{\pi y}{b}\right) \qquad (14)$$

Thus:

$$\mathbf{B} = -\nabla u = B_{y0}\left[\mathbf{a}_x\cos\left(\frac{\pi x}{b} - \alpha\right)\sinh\left(\frac{\pi y}{b}\right) + \mathbf{a}_y\sin\left(\frac{\pi x}{b} - \alpha\right)\cosh\left(\frac{\pi y}{b}\right)\right] \qquad (15)$$

The air-gap field is depicted in Fig. 3.

Let it be required to evaluate the torque generated about an axis parallel to the z-axis and passing through the point $P(x_0, y_0)$ in Fig. 2 using equation (1). Take S as the surface $y = 0$ which spans the coil. Note that, on S, $\mathbf{B}$ has only a y-component; hence $d\mathbf{M} \times \mathbf{B} = \mathbf{0}$ and the first integral in (1) yields nothing. To evaluate the contribution of the second integral in (1), first evaluate $(d\mathbf{M}.\nabla)\mathbf{B}$. On $y = 0$:

$$(d\mathbf{M}\cdot\nabla)\mathbf{B} = I\,L\,dx(\mathbf{a}_y\cdot\nabla)B = I\,L\,dx\frac{\partial \mathbf{B}}{\partial y} = \mathbf{a}_x I\,L\,dx B_{y0}\frac{\pi}{b}\cos\left(\frac{\pi x}{b} - \alpha\right) \qquad (16)$$

Therefore:

$$\mathbf{r}\times(d\mathbf{M}\cdot\nabla)\mathbf{B} = [\mathbf{a}_x(x - x_0) - \mathbf{a}_y y_0] \times \left[\mathbf{a}_x I\,L\,d x\mathbf{B}_{y0}\frac{\pi}{b}\cos\left(\frac{\pi x}{b} - \alpha\right)\right] = \mathbf{a}_z y_0 I\,L\,d x\mathbf{B}_{y0}\frac{\pi}{b}\cos\left(\frac{\pi x}{b} - \alpha\right) \qquad (17)$$

Substitution in (1) leads to

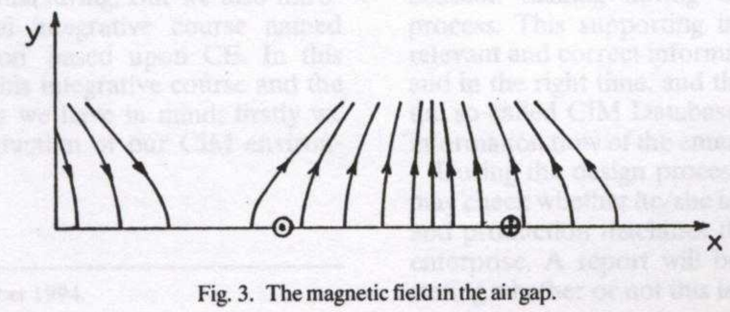

Fig. 3. The magnetic field in the air gap.

$$\mathbf{T} = \mathbf{a}_z y_0 I L \mathbf{B}_{y0} \frac{\pi}{b} \int_{x_0-pb/2}^{x_0+pb/2} \cos\left(\frac{\pi x}{b} - \alpha\right) \mathrm{d}x$$

$$= \mathbf{a}_z y_0\, I L\, \mathbf{B}_{y0}\, 2 \cos\left(\frac{\pi x_0}{b} - \alpha\right) \sin\frac{p\pi}{2} \quad (18)$$

As a check on this result, it is verified that the same result is obtained by other methods:

1. From Lorentz forces on the conductors:

$$T = I \int_C \mathbf{r} \times (\mathrm{d}\mathbf{r} \times \mathbf{B}) = -\mathbf{a}_y y_0(\mathbf{F}_1 + \mathbf{F}_2) \quad (19)$$

   where the integration is over the contour C of the circuit, and $\mathbf{F}_1$ and $\mathbf{F}_2$ are the Lorentz forces on the two conductors.
2. From change in flux linkage λ associated with a virtual angular displacement $\mathrm{d}\theta$ about an axis passing through the point P and parallel to the z-axis:

$$T = I \frac{\partial \lambda}{\partial \theta} = I\, y_0 \frac{\partial}{\partial x_0} \int_{x_0-pb/2}^{x_0+pb/2} B_y \,\mathrm{d}x \quad (20)$$

Details of these calculations are omitted.

CONCLUSION

In the foregoing, a simple derivation is provided for the torque acting on an electric circuit of any given shape when it is placed in a non-uniform magnetic field, in terms of magnetic moment, under the assumption that there are no currents on the surface S spanning the loop. The torque consists of two parts: (i) a couple, which is independent of the choice of reference point; and (ii) a part which can be considered to arise from forces of translation due to non-uniformity of the field and which depends on the choice of reference point. An example is given to show that the second term must be included (indeed, in the example considered, it is the only contributing term) for a correct evaluation of the torque.

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