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Developments in the Teaching of Mathematics to A-Level and the Implications for Engineering Courses*

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The paper describes the current developments in the teaching of mathematics in schools and colleges in England and Wales. These include the impact of the National Curriculum and the approaches to learning and content at advanced level such as those adopted within Nuffield Advanced Mathematics. Specific examples are presented to illustrate the methods of problem-solving that pupils develop in their courses. The implications for engineering teachers in higher education are considered. For example, lecturers need to be aware that significant numbers of future entrants will arrive with mathematical knowledge and skills that differ from those of the past. Also some of the new techniques for enhancing learning in schools will provide a good foundation for problem-based learning in engineering courses.

INTRODUCTION

THE British education scene is in the middle of substantial change at all levels. In schools and colleges the National Curriculum is being introduced to most courses for pupils up to the age of 16 years. Partly as a result of this, significant changes are also being made to A-level courses.

The aim of this paper is to summarize these changes in relation to mathematics and to discuss their likely effects on engineering courses.

NATIONAL CURRICULUM

All stated-funded schools and colleges in England and Wales are introducing the National Curriculum. Each subject, mathematics, English, science, etc., has its own National Curriculum. Not all subjects are being introduced at the same time. The core subjects, mathematics, English and science will be completed by 1994.

In mathematics, the National Curriculum is divided into five target areas.

1. Using and applying mathematics including applications, communication, reasoning, logic and proof.
2. Numbers: knowledge and use of numbers, estimation and measures.
3. Algebra: patterns, formulas, equations, inequalities and graphs.
4. Shape and space: shape, location, movement and measures.
5. Handling data: collecting, processing, representing, interpreting and probability.

Each target is broken down into ten levels, with 5 year olds starting at level 1. By the age of 16 years a small percentage of pupils will have achieved level 10, another small percentage will have achieved no more than level 4, but the majority will be spread out in between. Levels 7–10 roughly correspond to GCSE grades A–C or the old O-level pass.

For example, at level 8 a pupil needs to be able to manipulate algebraic formulae, equations or expressions, as well as design and use a questionnaire or experiment to test a hypothesis.

At the same time, in information technology pupils will have used computers to explore possible uses of spreadsheets and databases.

Pupils will be assessed at four key stages (KS): KS1 (7 year olds), KS2 (11 year olds), KS3 (14 year olds) and KS4 (16 year olds). The assessment is mainly based on pencil and paper tests, with some other work (course work) set and marked by teachers with external moderation. Marks for course work at KS4 may not exceed 20% of the total.

One of the main advantages of the mathematics National Curriculum is that similar ability 16 year olds throughout the country will have covered the same work. However, the introduction of testing has been fraught with mismanagement and controversy, focusing on the enormous administrative demands on teachers and on the publication of individual school/college results.

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DEVELOPMENTS POST-16 YEARS

There is much development in course content, teaching and assessment post-16 years. This development is essentially a result of the introduction of the National Curriculum pre-16 years. However, many students and teachers also feel dissatisfaction with the traditional focus on two 3 h examinations over-full with algebraic manipulation, at the expense of other less abstract areas of work.

In September 1994, 16 year old students (approximately 70 000 each year, half from schools and half from colleges) will begin new A-level courses, because all present courses need to be changed to take account of the National Curriculum pre-16 years. The courses will be either A-level or AS-level (half the content but the same depth as an A-level). Students could take 3 A-levels or 2 A-levels and 2 AS-levels or 3A-levels and 1 AS-level etc.

In terms of content, there will be syllabuses which still offer a choice of pure and mechanics or pure and statistics. However, many students will have been exposed to a mixture of all branches of mathematics with the option of focusing on areas such as Newtonian mechanics or statistics, but also on others such as algorithms or mathematics and the arts. These courses will often be presented and assessed in a modular form, again with a maximum of 20% course work.

Many new courses such as Nuffield and SMP take advantage of technology to develop and explore mathematics in a modelling context rather than solely as an abstract algebraic exercise. For example, the new Nuffield Advanced Mathematics takes full advantage of the appropriate use of widely used technology such as graphics calculators to complement the algebraic work which all syllabuses must still retain. The graphical, algorithmic and statistical facilities of graphic calculators are a powerful aid and back-up for students. The technology is widely available, at the price of a pair of training shoes and generally students are prepared to purchase their own calculator.

EXAMPLES (A-LEVEL MATHEMATICS)

To give a flavour of one aspect of the ways in which a graphics calculator can be used, we have taken some work from the Nuffield Advanced Mathematics course.

In the Nuffield course, students will develop their ability to solve equations such as $x^2 - 3x + 1 = 0$ using and comparing several methods.

1. Quadratic formula $\quad x = \dfrac{3 \pm \sqrt{3^2 - 4 \times 1 \times 1}}{2 \times 1}$
2. Completing the square $(x - 1.5)^2 - 2.25 + 1 = 0$.
3. Draw the graph of $f(x) = x^2 - 3x + 1$ then zoom and trace on the intercepts.
4. Use a bisection algorithm:
(Comment: L is lower bound, H is higher bound)

   Input L, H and an equation $f(x) = 0$
   repeat
   $\frac{1}{2}(L + H) \rightarrow M$
   if $f(L) \times f(M) > 0$
   then $M \rightarrow L$
   else $M \rightarrow H$
   until $H - L < 0.00001$
   Output M
5. Use a fixed-point iteration such as $x_{n+1} = \sqrt{(1 - 3x_n)}$ which can be illustrated using a 'cobweb diagram' which can easily be incorporated into a calculator program and graph display.
6. Use the Newton–Raphson iteration in a similar way.

$$x_{n+1} = x_n - \frac{x_n^2 - 3x_n + 1}{2x_n - 3}$$

We hope that this small example illustrates the fact that many areas of mathematics can be made visual and, hence, accessible to a wider range of students. The graph plotting facilities have many uses in addition to simple curve sketching.

A prominent theme in the Nuffield course is modelling. Examples take students through stages of model definition, simplification and refinement.

ENTRANTS TO ENGINEERING COURSES

How will holders of mathematics A-level, as they become entrants to university engineering degrees, differ from those of the past?

A trend that lecturers have been aware of for many years is a reduction in the coverage of mechanics. The emphasis in some pure and applied A-level papers has moved away from mechanics, but the main problem is that fewer pupils have been choosing mechanics, when given a choice between mechanics and statistics.

This trend is likely to continue. In the new A-levels, all candidates will be required to take some statistics and will therefore have less time for mechanics. Other areas of mathematics such as discrete mathematics (critical path, etc.), algorithms, coding and problem solving will also be offered alongside mechanics and statistics, further diluting the mechanics content. As already described, many courses will be modular; mechanics will be included, but will not, on the basis of present trends, be favoured. Also it is worth noting that mechanics syllabuses will give more emphasis to the modelling theme, including theory/practical loops. Newtonian mechanics is likely to be part of all A-level schemes, but inevitably a reduced proportion. (It is also relevant to note that the number of students taking physics at A-level is currently dropping dramtically.)

Many mathematics A-level courses will place emphasis on encouraging pupils to explore the applications of mathematics and will give accordingly less emphasis to rote learning of advanced techniques of algebraic manipulation.

In a group of successful A-level mathematics students there will be a wider variety of subject knowledge than in the past. This will have resulted in part from the modular systems and from the differences between the A-level syllabuses available. There will also be a wider experience of learning methods. Some will be used to traditional teaching techniques, but many others will be used to taking more initiative in learning. Many will own a graphics calculator and be familiar with its power as a learning aid. All will be familiar with technology such as databases and spreadsheets.

In addition, many entrants will have had a wider sixth form education than in the past by taking AS-levels. A-level combinations will continue to be increasingly diverse (for example, mathematics, German and psychology).

CONSEQUENCES

Mathematics A-level holders of the future will be different from those of the past. Lecturers should not dwell on what they see as the negative aspects. Spotting 'gaps' in the mathematical knowledge of entrants has been a popular preoccupation for years. Pollard [1], for example, found in a 1989 survey that engineering departments 'mentioned a variation in the mathematics knowledge of new entrants and a lack of knowledge in some areas of mathematics where knowledge would previously have been assumed'.

The positive side is that these entrants are likely to be better prepared for and more receptive to problem-based approaches to learning. They may have less inclination to deal with mathematical concepts in the abstract, but may be more likely to understand and develop material in context. They are likely to be familiar with problem-based approaches from their secondary education and confident in using technology to help them learn for themselves. This should equip them well for engineering courses in which the value of problem-based learning, project work and other student-led activities is becoming increasingly recognized [2].

Many consider that moving away from traditional undergraduate engineering timetables, heavy with formal lecturing, can not only enhance the quality of learning but also encourage more young people to study engineering. The benefits will be even greater if the new approaches in secondary mathematics reverse present trends and encourage more pupils to continue to A-level.

Lecturers may be able to find fault with the new systems in schools and the new types of university entrants they produce, but it is of prime importance that they fully understand the nature of the changes and adapt their approaches where appropriate in order to exploit the new possibilities.

EXAMPLES (ENGINEERING COURSES)

As already described, one consequence of the new mathematics A-levels will be that increasing numbers of entrants to engineering courses will be owners of graphics calculators. As many students and lecturers are discovering, these have considerable potential for enhancing engineering studies. Opportunities for exploiting their power as learning aids exist throughout engineering courses, for example, in engineering science, in design, in laboratory classes and in engineering mathematics.

The example below is of material that could be used to allow part of an engineering mechanics subject to be learnt through experiment using a graphics calculator. It concentrates simply on use of graph-plotting and programming facilities (a small proportion of the functions available). The benefits of the approach are likely to be increased if students have been familiar with this style of learning from a younger age. The example is of hydraulic analysis of a simple surge shaft.

Figure 1 shows a simplified arrangement for a surge shaft on a hydroelectric scheme. Friction losses along the tunnel are represented by h_f.

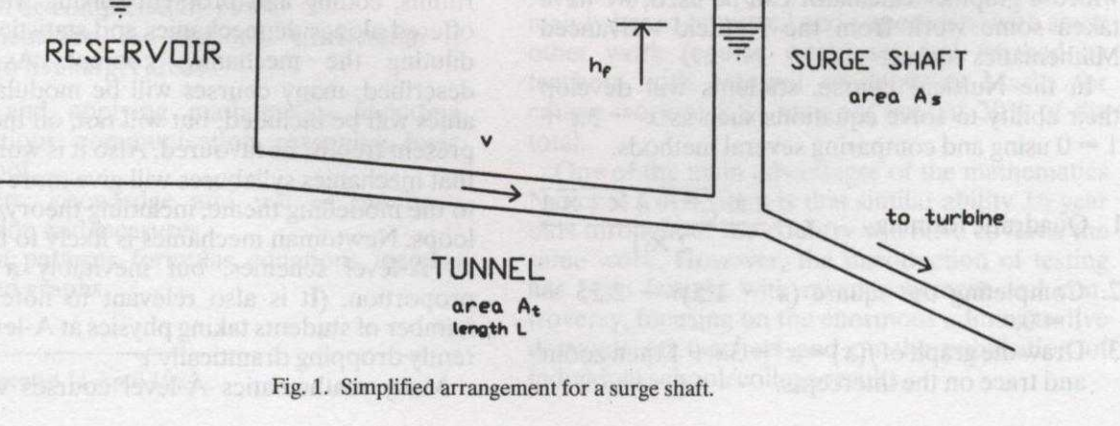

Fig. 1. Simplified arrangement for a surge shaft.

Initially, the level in the surge shaft is constant at $y = -h_f$ and velocity in the tunnel is constant at v. We can simplify the relationship between h_f and v as $h_f = kv^2$ where k is a constant.

If flow to the turbine is suddenly stopped, water rises up the surge shaft. This causes a reversal of flow in the tunnel, leading to a series of oscillations. Of particular interest (in the design of a system) is the maximum level of water in the surge shaft. A dynamic equation (Eqn. 1) and a continuity equation (Eqn. 2) can be derived ('model definition'):

$$\frac{l}{g}\frac{dv}{dt} + y + kv|v| = 0 \qquad (1)$$

$$A_T v = A_s \frac{dy}{dt} \qquad (2)$$

where t = time.

Exercise

DATA: $A_T = 5\ \text{m}^2$
$A_s = 25\ \text{m}^2$
$l = 1500\ \text{m}$
initial $v = 3\ \text{m/s}$

1. First assume that friction losses in the tunnel are negligible ('model simplification'), i.e. $k = 0$ and initial $y = 0$. This simplifies the solution and provides a 'worst case' value of maximum level (since friction will have a dampening effect). Equation 1 becomes

$$\frac{l}{g}\frac{dv}{dt} + y = 0$$

From Eqn. 2

$$v = \frac{A_S}{A_T}\frac{dy}{dt},$$

so

$$\frac{dv}{dt} = \frac{A_S}{A_T}\frac{d^2y}{dt^2}$$

therefore

$$\frac{l}{g}\frac{A_s}{A_T}\frac{d^2y}{dt^2} + y = 0$$

The solution of this equation, for initial conditions $t = 0$, $y = 0$, $v = v_0$ is

$$y = v_0\sqrt{\frac{l}{g}\frac{A_T}{A_s}}\sin\frac{2\pi}{T}t$$

where the period of oscillation

$$T = 2\pi\sqrt{\frac{l}{g}\frac{A_S}{A_T}}$$

Plot on your calculator the variation of water level in the surge tank with time. Determine the maximum level and the time at which it occurs. Check with expressions for period, and amplitude

$$v_0\sqrt{\frac{l}{g}\frac{A_T}{A_S}}$$

Positive v represents flow towards the surge shaft and negative v represents flow towards the reservoir. What will be happening to the value of v when y has its maximum value? When y has its minimum value? (Think about the physical meaning of Eqn. 2.)

2. If friction is included, a simple way of solving Eqs 1 and 2 is by considering time steps of Δt. During each Δt, the value of y varies by Δy, and v by Δv. This method is accurate only if Δt is small.

Equations 1 and 2 can be rewritten as

$$\frac{l}{g}\frac{\Delta v}{\Delta t} + y + kv|v| = 0$$

$$A_T v = A_S \frac{\Delta y}{\Delta t}$$

We will know the values of v and y at the start of each time step. We then determine Δv and Δy using

$$\Delta v = \frac{g\Delta t}{l}[-y - kv|v|]$$

$$\Delta y = \frac{A_T}{A_S}v\Delta t$$

These will allow us to calculate v and y at the end of the time step ($y_{end} = y_{start} + \Delta y$). y_{end} becomes y_{start} for the next time step. We must now have information about friction losses in addition to the DATA already given. In this case $k = 1$, therefore initial $y = -9$ m. Use a program on your calculator to investigate the variation of v and y with time. Make $\Delta t = 10$ s. Write the program yourself or use the one below.

```
10 → D
3 → V
−9 → Y
0 → T
Lbl 1
    Disp 'T V Y'
    Disp T
    Disp V
    Disp Y
    Pause
    9.81 * D * (−Y − 1 * V * abs V)/1500 → W
    5 * V * D/25 → Z
    V + W → V
    Y + Z → Y
    T + D → T
If T ≤ 100
Goto 1
```

Estimate the maximum level in the surge tank and the time at which it occurs. Note the relationship beween the variations of y and the variations of v. Now reduce the size of the time step ('model refinement'); make $\Delta t = 1$ s. Again estimate the maximum level. Which is the more accurate estimate? Investigate the influence of other parameters: first k, then any of the others. Write a program, for the solution including friction, which plots the variation of y with time.

CONCLUSIONS

The current changes in mathematics teaching in schools and colleges are not likely to close the 'gaps' in mathematical knowledge which engineering lecturers have been spotting for years in entrants to their courses. They are however likely to increase receptiveness to problem-based approaches to learning, including use of graphics calculators. Lecturers should make themselves aware of the changes in order to be prepared for the shortcomings and exploit the opportunities.

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John Davies took a BSc, followed by an MSc, in civil engineering, at University College of Swansea. He worked as a civil engineer for 8 years, first as a graduate engineer for consulting engineers, then as a chartered engineer for a water authority. In 1980 he became a lecturer in civil engineering at the Polytechnic of Central London (now University of Westminster). His main research interest is in hydraulics of sewer systems; he completed a PhD in 1990. He is currently principal lecturer and course leader of the BEng in civil engineering.

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