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On Teaching Establishing the Existence of Limit Cycles*

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The Lyapunov stability theorems and their use in establishing the existence of limit cycles have been taught on final-year undergraduate control theory/engineering courses in the UK for many years. However, the application of the theorems in this context is relatively less elegant and complete than that of the more sophisticated invariant set theorems. In this paper the advantages of using the invariant set theorems, as against the Lyapunov stability theorems, in teaching establishing the existence of limit cycles are highlighted.

INTRODUCTION

THE LYAPUNOV stability theorems and their use in establishing the existence of limit cycles have been taught on final-year undergraduate control theory/engineering courses in the UK for many years. However, the Lyapunov theorems address the stability of equilibrium points, taken by change of variable, to lie at the origin of state space.

For a stable limit cycle the system settles down to cyclical behaviour at the output. The Lyapunov theorems are of interest with respect to establishing the existence of limit cycles in that they introduce the concept of a Lyapunov function as a measure of the total energy of the system considered. Essentially, if starting from within a region around the origin the derivative with respect to time of the Lyapunov function is always negative, then within that region total energy must eventually decay to zero. It follows that the system is asymptotically stable within that region. This use of the concept of a Lyapunov function has, traditionally been intuitively extended to address the establishing of the existence of limit cycles. Noting that a system exhibiting cyclical behaviour at the output is 'stuck' at a constant energy level, it is argued that it follows that the derivative with respect to time of the Lyapunov function will be zero starting from all points corresponding to the energy level of the limit cycle. Limit cycles are, therefore, identified from the solutions to the equation obtained by setting the derivative of the Lyapunov function with respect to time to zero.

All points corresponding to the energy level of a limit cycle form a closed trajectory (an invariant set) in state space, as should be clear from the associated cyclical behaviour at the output. Consequently, stable limit cycle behaviour is best understood as being associated with the system converging to an invariant set within state space. The invariant set theorems specifically address the problem of determining whether a system converges to an invariant set in state space.

In the 1991/92 academic year the invariant set theorems and their application (details taken from Slotine and Weiping Li [1]) were taught and examined, through a single question, for the first time on the control and systems theory course for final-year engineering undergraduates at Brunel University. In this paper, following presentation of the mathematical background, the examination question concerned is presented together with the corresponding question from the previous paper. The questions are followed by relevant extracts from their solutions. Students' examination performance over successive years is then reported. The conclusions drawn from the preceding are then presented.

MATHEMATICAL BACKGROUND

Let B_R denote the spherical region (or ball) defined by $\|x\| < R$.

Definition. A scalar continuous function $V(\mathbf{x})$ is said to be *locally positive definite* if $V(\mathbf{0}) = 0$ and, in a ball B_R $\mathbf{x} \neq \mathbf{0}$ $V(\mathbf{x}) > 0$.

Definition. If, in a ball B_R, the function $V(\mathbf{x})$ is positive definite and has continuous partial derivatives, and if its time derivative along any state trajectory of an autonomous system of the form $\dot{\mathbf{x}} = f(\mathbf{x})$, is negative semi-definite, i.e. $\dot{V}(\mathbf{x}) \leqslant 0$, then $V(\mathbf{x})$ is said to be a *Lyapunov function* of the system.

Theorem (*Lyapunov's theorem for local stability*). If, in a ball B_R, there exists a scalar function with continuous first partial derivatives such that

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$V(\mathbf{x})$ is positive definite (locally in B_R)
$\dot{V}(\mathbf{x})$ is negative semi-definite (locally in B_R)

then the equilibrium point **0** is stable. If, in fact, the derivative $\dot{V}(\mathbf{x})$ is locally negative definite in B_R, then the stability is asymptotic.

Theorem (*local invariant set theorem*). Consider an autonomous system of the form $\dot{\mathbf{x}} = f(\mathbf{x})$ with f continuous, and let $V(\mathbf{x})$ be a scalar function with continuous first partial derivatives. Assume that

for some $\iota > 0$, the region Ω_ι defined by $V(\mathbf{x}) < \iota$ is bounded
$\dot{V}(\mathbf{x}) \leqslant 0$ for all $\mathbf{x}$ in Ω_ι.

Let R be the set of all points within Ω_ι where $\dot{V}(\mathbf{x}) = 0$, and M be the largest invariant set in R. Then, every solution $\mathbf{x}(t)$ originating in Ω_ι tends to M as $t \to \infty$.

SAMPLE EXAMINATION QUESTIONS

Bachelor of Engineering Degree Examination
Electrical Engineering and Electronics (Honours)
Part II
EE 451 Control and Systems Theory
Time allowed—3 hours
Four questions to be attempted

June 1991–Question 7

State the second (direct) stability theorem of Lyapunov.

Explain how a Lyapunov function may indicate the existence of a stable limit cycle. What is the significance of the existence of a stable limit cycle in the response of a non-linear system to non-zero initial conditions?

(i) If a non-linear system is described by the equations:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_1^3 - x_2$$

show that the only equilibrium point of the system lies at the origin of the state-space. Using a Lyapunov function of the form:

$$V(\mathbf{x}) = 2x_1^2 + x_1^4 + 2x_2^2$$

show that the origin is globally asymptotically stable.

(ii) Show that the non-linear system:

$$\dot{x}_1 = -x_2 + x_1(1 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = x_1 + x_2(1 - x_1^2 - x_2^2)$$

has a limit cycle, and determine its nature.

June 1992—Question 2

Define an **invariant set** for a dynamic system.

State, without proof, the global invariant set theorem.

Limit cycles are usually undesirable in control systems. Give **three** reasons why this may be so.

Show that the non-linear system:

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 10)$$
$$\dot{x}_2 = -x_1 - 3x_2^3(x_1^2 + x_2^2 - 10)$$

has a limit cycle and determine its nature.

RELEVANT EXTRACTS FROM THE SOLUTIONS

June 1991—Question 7

A Lyapunov function such that $\dot{V}(\mathbf{x}) = 0$ for some closed trajectory indicates the existence of a *limit cycle*.

Taking $V(\mathbf{x}) = 0.5x_1^2 + 0.5x_2^2$

$$\dot{V}(\mathbf{x}) = x_1\dot{x}_1 + x_2\dot{x}_2$$

The non-linear system

$$\dot{x}_1 = -x_2 + x_1(1 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = x_1 + x_2(1 - x_1^2 - x_2^2)$$

gives

$$\dot{V} = (x_1^2 + x_2^2)(1 - x_1^2 - x_2^2)$$
$$\dot{V} = 0 \text{ for } x_1^2 + x_2^2 = 1$$

i.e. a circle of radius 1 centred at the origin.

If $x_1^2 + x_2^2 > 1$, since $(x_1^2 + x_2^2) > 0$ always, $\dot{V} < 0$
If $x_1^2 + x_2^2 > 1$, since $(x_1^2 + x_2^2) > 0$ always, $\dot{V} > 0$

So starting from initial conditions inside the unit circle the system energy increases until the limit cycle trajectory is reached. Starting from initial conditions outside the unit cycle system energy decreases until the limit cycle trajectory is reached. Therefore, the limit cycle is *stable*.

June 1992—Question 2

A set G is an *invariant set* for a dynamic system if every system trajectory which starts from a point in G remains in G for all future time.

First, notice that the set defined by $x_1^2 + x_2^2 = 10$ is invariant, since

$$\frac{\mathrm{d}(x_1^2 + x_2^2 - 10)}{\mathrm{d}t} = 2x_1\dot{x}_1 + 2x_2\dot{x}_{2\,1}\dot{x}_1 + 2x_2\dot{x}_2$$
$$= -(2x_1^2 + 6x_2^2)(x_1^2 + x_2^2 - 10)$$

which is zero on the set. The motion *on* this invariant set is described (equivalently) by *either* of the equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$

Therefore, we see that the invariant set actually represents a *limit cycle*.

To see if this limit cycle is actually attractive, let us define as a Lyapunov function candidate

$$V = (x_1^2 + x_2^2 - 10)^2$$

Using our earlier calculations, we immediately obtain:

$$\dot{V} = -4(x_1^2 + 3x_2^2)(x_1^2 + x_2^2 - 10)^2$$

$\dot{V}$ is strictly negative except if

$$x_1^2 + 3x_2^2 = 0, \quad \text{or} \quad x_1^2 + x_2^2 = 10$$

The first equation is verified only at the origin. The second equation is simply that defining the limit cycle.

The equilibrium point at the origin can be shown to be *unstable*.

Consider the region Ω_{100} (i.e. x_1, x_1 such that $V(x_1, x_2) < 100$), and note that while the origin does not belong to Ω_{100}, every other point in the region enclosed by the limit cycle is in Ω_{100} (in other words, the origin corresponds to a local *maximum* of V). Thus, while the expression of $\dot{V}$ is the same as before, now the invariant set is just the limit cycle. Therefore, the application of the local invariant set theorem shows that any state trajectory starting from the region within the limit cycle, excluding the origin, actually converges to the limit cycle. In particular, this implies that the equilibrium point at the origin is unstable.

EXAMINATION PERFORMANCE

June 1991—Question 7

Number of candidates taking the examination:	28
Number of candidates attempting question 7:	24
Percentage of candidates attempting question 7:	85.7%
Average mark obtained by the candidates on question 7:	13.2
Average percentage of available marks on the question obtained:	52.7%

June 1992—Question 2

Number of candidates taking the examination:	16
Number of candidates attempting question 2:	16
Percentage of candidates attempting question 2:	100%
Average mark obtained by the candidates on question 2:	15.8
Average percentage of available marks on the question obtained:	63.3%

CONCLUSIONS

In teaching establishing the existence of limit cycles, the conceptual advantages of using the invariant set theorems rather than the Lyapunov stability theorems have been demonstrated. This has been done through the presentation of past examination questions and extracts of the solutions to the questions.

On examining the use of the invariant set theorems in establishing the existence of limit cycles, it was found that a higher percentage of the students taking the Control and Systems Theory examination attempted the category of question concerned than in previous years, when the use of the Lyapunov stability theorems in establishing the existence of limit cycles had been taught. Furthermore, the average mark obtained on the category of question was higher than before. Of course, these observations do not constitute an evaluation of the curriculum change made but do indicate that the change did not degrade students' level of attainment under examination conditions.

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REFERENCE

1. J. J. E. Slotine and Weiping Lei (1990) *Applied Nonlinear Control*. Prentice Hall, Englewood Cliffs, NJ.

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