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A One-dimensional Combined-change Model for Compressible Flow*

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An analytic combined-change one-dimensional compressible flow model is presented which: includes all simple changes as special flow cases; extends the applicability of simple change models to certain combined-change flow cases; provides an analytic means of solving combined-change problems; illustrates the effects of normal shocks; and determines the processes for which a wide array of properties are constant. Properties examined include Mach number, velocity, speed of sound, density, static pressure and temperature, stagnation pressure, and entropy. One case of constant Mach number is shown to locate the sonic point for combined-change flow through a converging–diverging nozzle.

1. The paper discusses material which can be used in the following courses:
   Introductory gas dynamics or similar course.
2. Students of the following departments may benefit from the course/discussion in the paper:
   Aeronautical engineering and mechanical engineering.
3. Level of the course:
   Senior and/or introductory graduate level.
4. Mode of presentation:
   Traditional lecture plus computer software that students can use to do homework for the entire course.
5. Is the material presented in a regular or elective course?
   The course is usually required for aeronautical engineers and may be required or elective for mechanical engineers.
6. Class hours required to cover the material:
   1–2 hours depending on the depth of coverage desired. This material is intended to augment the coverage of combined changes once the constitutive equations are developed.
7. Student homework and revision hours required for the materials:
   The material would take 1.5–3 additional hours of homework; however, the computer software available with this paper would cut at least that amount of time from other homework assignments just through the elimination of tabular interpolation.
8. Brief description of novel aspects of the paper:
   The compressible flow model presented is new. The model presents the opportunity for teachers to build on the understanding they have developed in their students for simple changes and extend this understanding to combined-change flows. The model has also been shown to provide new information for constant property flows and sonic point location that considerably expands that available in standard texts such as Shapiro (Chapter 8), Saad (Chapter 6), or Zucrow and Hoffman (Chapter 9).
9. The standard text recommended in addition to the author's notes:
   See any of the above. While the paper itself is only applicable to the portion of compressible flow in the chapters above, the computer software is applicable to the entire course.
10. Is the material covered in the text? In what way is the text discussion different from the paper?
   The paper describes an analytic model for combined-change problems which had previously been approachable only with numerical approximations. Further, since the model is analytic it permits the association of combined-change flow behavior with that of simple change flows which is impractical using numerical approximations.

* Paper accepted 10 April 1993.

NOMENCLATURE

A	area
C	change coefficients
C_1	combination of change coefficients given by Eq. (5)
C_2	combination of change coefficients given by Eq. (6)
c_v	specific heat at constant volume
D	hydraulic diameter

F	generalized independent variable
Fxy	generalized independent variable ratio across a normal shock
f	friction coefficient
ff	function of parametric variables
g	static pressure mass flow function
k	specific heat ratio
L	length
M	Mach number
p	static pressure
P	isentropic stagnation pressure
s	entropy
T	temperature
V	velocity
w	mass flow rate
x	distance in the direction of flow
y	ratio of the x component of injected flow velocity to the main flow velocity
ρ	density

SUBSCRIPTS

a	pertaining to area change
cc	pertaining to a combined-change flow
f	pertaining to fluid friction
$isen$	pertaining to isentropic flow
$inj0$	pertaining to simple injection with $y = 0$
$inj1$	pertaining to simple injection with $y = 1$
Ray	pertaining to Rayleigh flow
0	isentropic stagnation state or injection with $y = 0$
1	pertaining to injection with $y = 1$
t	pertaining to heat transfer
w	pertaining to mass flow injection

SUPERCRIPTS

*	at $M = 1$
′	non-dimensionalized

INTRODUCTION

CLASSIC analysis of one-dimensional, steady, compressible flow of perfect gases is well documented by a number of authors; see: Shapiro [1], Chapman and Walker [2], Zucrow and Hoffman [3], and Saad [4]. The classic approach examines the simplest case of each of the usual independent variables of compressible flow: area change, heat transfer, friction and mass injection. From a pedagogical point of view, combined-change analysis should represent a generalization of simple changes. While in theory, it does provide this generalization, many students fail to make the connection.

Shapiro, for example, presents the traditional approach to combined-change problems which begins with the constitutive differential equations, considers limited two-independent-variable cases, and then primarily uses numerical procedures for the more general case. Beans [5] and Hodge [6] provide computer programs for the numerical solution of combined-change problems. Zucrow and Hoffman [3] devote a chapter to generalized, steady, one-dimensional flow. In addition to the numerical procedures, they consider two analytic cases involving constant Mach number. Unless students expend substantial time in numerically analyzing combined-change processes, they are unlikely to see any similarities in combined-change and simple change flow behavior.

Young [7] presents an analytic combined-change model for steady one-dimensional flow of perfect gases. This model provides an analytic tool for gaining insights into the behavior of combined-change flows. Simple change behavior can be, through this model, associated with a range of combined-changes as a logical means of building on students' understanding.

COMBINED-CHANGE MODEL

In the combined-change model presented by Young [7], each of the independent variables is represented by a constant change coefficient, C, and a generalized variable, F, as shown below.

$$\frac{dA}{A} = C_a \frac{dF}{F} \tag{1}$$

$$\frac{dT_0}{T_0} = C_t \frac{dF}{F} \tag{2}$$

$$\frac{4f dx}{D} = C_f \frac{dF}{F} \tag{3}$$

$$\frac{dw}{w} = C_w \frac{dF}{F} \tag{4}$$

Additional definitions are made for convenience.

$$C_1 = 2C_w + C_t - 2C_a \tag{5}$$

$$C_2 = C_t + C_f + 2C_w(1 - y) \tag{6}$$

These relations define the thermodynamic path of the process. Since combined-change flow is path-dependent, the model is not completely general. The model will be shown to be sufficiently robust to give new insight into the combined-change flow.

The generalized variable is usually, except in the case of simple friction, identified as one of the formerly independent variables in a specific analysis. Substituting Eqs (1)–(4) into the constitutive equation for Mach number and integrating from $M = 1$ to M, an expression for the generalized variable ratio, F/F^*, is obtained. The constitutive equations for Mach number and all other parameters to be held constant are taken from Shapiro [1].

For $C_1 \neq 0$

$$F' = \left(\frac{F}{F^*}\right)^{C_1} = M^2 ff\left(M, k, \frac{C_2}{C_1}\right) \tag{7}$$

The function $ff(M, k, C_2/C_1)$ is shown below

$$ff\left(M, k, \frac{C_2}{C_1}\right) = \left\{ \left[\frac{(k+1)}{(2+(k-1)M^2)}\right]^{(k+1)} \left[\frac{\left(1+kM^2\frac{C_2}{C_1}\right)}{\left(1+k\frac{C_2}{C_1}\right)}\right]^{2\left(1+k\frac{C_2}{C_1}\right)} \right\}^{\frac{1}{k-1-2k\frac{C_2}{C_1}}}$$

$$T' = \frac{\frac{T}{T^*}}{\frac{T_0}{T_0^*}} = \frac{(k+1)}{[2+(k-1)M^2]} \tag{8}$$

Note that T_0/T_0^* could be written $(F/F^*)^{C_t}$ as in the equation for s' below.

$$p' = \frac{p}{p^*} = \sqrt{\frac{(k+1)}{[2+(k-1)M^2]}\, ff\left(M, k, \frac{C_2}{C_1}\right)} \tag{9}$$

$$P_0' = \frac{P_0}{P_0^*} = \sqrt{ff\left(M, k, \frac{C_2}{C_1}\right)\left[\frac{(k+1)}{(2+(k-1)M^2)}\right]^{\frac{-(k+1)}{(k-1)}}} \tag{10}$$

$$s' = \frac{(s-s^*)}{c_v} - kC_t \ln\frac{F}{F^*}$$

$$s' = \ln\left[\sqrt{\frac{(k+1)^{(k+1)}}{(2+(k-1)M^2)^{(k+1)}\, ff\left(M, k, \frac{C_2}{C_1}\right)^{(k-1)}}}\right] \tag{11}$$

For $C_1 = 0$

$$\left(\frac{F}{F^*}\right)^{C_2} = \left[\frac{2\left(1+\frac{(k-1)}{2}M^2\right)}{(k+1)M^2}\right]^{\frac{(k+1)}{2k}} \exp\frac{-(1-M^2)}{kM^2} \tag{12}$$

$$p' = \frac{p}{p^*} = \sqrt{\frac{k+1}{2M^2\left(1+\frac{(k-1)}{2}M^2\right)}} \tag{13}$$

$$P_0' = \frac{P_0}{P_0^*} = \frac{1}{M}\sqrt{\left[\frac{(2+(k-1)M^2)}{(k+1)}\right]^{\frac{(k+1)}{(k-1)}}} \tag{14}$$

$$T' = \frac{\frac{T}{T^*}}{\frac{T_0}{T_0^*}} = \frac{k+1}{[2+(k-1)M^2]} \tag{15}$$

$$s' = \frac{(s-s^*)}{c_p} - C_t \ln\frac{F}{F^*}$$

$$s' = \ln\left\{M^2\sqrt{\left[\frac{k+1}{M^2(2+(k-1)M^2)}\right]^{\frac{k+1}{k}}}\right\} \tag{16}$$

Young [7] also demonstrates how the functions above can be used numerically to solve simple and combined-change problems.

On request, the authors will supply a complimentary copy of a PC-based compressible flow calculator which calculates all the functions above and their inverses as well as most other one-dimensional functions.

SIMPLE CHANGES

The change coefficients in Eqs (1)–(6) and the constants C_1 and C_2 can be readily determined for each of the simple change flow types. For example, for isentropic or simple area change flow the generalized variable, F, may be chosen as the area, A. Then from Eqs (1)–(4) the change coefficients may be found.

$$C_a = 1 \quad C_t = 0 \quad C_f = 0 \quad C_w = 0$$

Then C_1, C_2 and C_2/C_1 can be determined.

$$C_1 = 2(0) + 0 - 2(1) = -2$$

$$C_2 = 0 + 0 + 2(0)(1-y) = 0$$

$$\frac{C_2}{C_1} = \frac{0}{-2} = 0$$

Table 1 contains a list of the change coefficients, determined in a similar manner, for each of the usual simple change flows. There are a number of observations that can be made from Table 1. For simple injection with $y = 1$, the flow functions from Eqs (7)–(11) are the same as for isentropic flow. For simple injection with $y = 0$, the flow functions are the same as for Rayleigh flow. For simple change flows, integration of Eqs (1)–(4) yields the relationships on the first line below while substitutions of the first line into Eqs (7) or (12), depending on the value of C_1, yields the expression for F' on the second line.

Simple change relations

Isentropic

$$\left(\frac{F}{F^*}\right)^{C_a} = \frac{F}{F^*} = \frac{A}{A^*}$$

$$F' = \left(\frac{F}{F^*}\right)^{C_1} = \left(\frac{A}{A^*}\right)^{-2}$$

Rayleigh

$$\left(\frac{F}{F^*}\right)^{C_t} = \frac{F}{F^*} = \frac{T_0}{T_0^*}$$

$$F' = \left(\frac{F}{F^*}\right)^{C_1} = \frac{T_0}{T_0^*}$$

Table 1. Change coefficients for simple-change flows

Simple change	C_a	C_t	C_f	C_w	C_1	C_2	C_2/C_1
Isentropic	1	0	0	0	−2	0	0
Rayleigh	0	1	0	0	1	1	1
Fanno	0	0	1	0	0	1	∞
Injection	0	0	0	1	2	$2(1-y)$	$1-y$

Fanno

$$\left(\frac{F}{F^*}\right)^{C_f} = \frac{F}{F^*} = \exp\frac{4fL}{D}$$

$$F' = \left(\frac{F}{F^*}\right)^{C_2} = \exp\left(\frac{4fL}{D}\right)$$

Injection

$$\left(\frac{F}{F^*}\right)^{C_w} = \frac{F}{F^*} = \frac{w}{w^*}$$

$$F' = \left(\frac{F}{F^*}\right)^{C_1} = \left(\frac{w}{w^*}\right)^2$$

Therefore, the statement that injection with $y = 0$ has the same set of flow functions as Rayleigh flow implies:

$$F'_{Ray} = F'_{inj0} = \frac{T_0}{T_0^*} = \left(\frac{w}{w^*}\right)_0^2$$

and for $y = 1$:

$$F'_{isen} = F'_{inj1} = \left(\frac{A}{A^*}\right)^{-2} = \left(\frac{w}{w^*}\right)_1^2$$

or

$$\left(\frac{w}{w^*}\right)_1 = \frac{1}{\dfrac{A}{A^*}}$$

These results enable simple injection flow problems with $y = 0$ or $y = 1$ to be solved utilizing Rayleigh or isentropic flow functions, respectively.

COMBINED CHANGES

This line of reasoning can be extended a step further to apply simple change flow functions to combined-change problems. Examination of Eqs (1)–(6) shows that while a certain set of values for the change coefficients gives unique values of C_1 and C_2, the reverse is not true. In general, there are an infinite number of combinations of the change coefficients that can give the same value of C_1 or C_2. For example, suppose that $C_a = 0.25$, $C_t = 1$, $C_f = -0.5$, and $C_w = 0.5$ with $y = 0$, then the values of C_1 and C_2 may be calculated to be 1.5 and 1.5 respectively. Therefore, the value of C_2/C_1 is 1, the same as for Rayleigh flow. The relationship among the change variables can then be developed in a manner analogous to that above for simple changes.

$$\left(\frac{F}{F^*}\right)^{C_t} = \frac{F}{F^*} = \left(\frac{T_0}{T_0^*}\right)_{cc}$$

$$\left(\frac{F}{F^*}\right)^{C_a} = \left(\frac{F}{F^*}\right)^{0.25} = \left(\frac{T_0}{T_0^*}\right)_{cc}^{0.25} = \frac{A}{A^*}$$

$$\left(\frac{F}{F^*}\right)^{C_w} = \left(\frac{T_0}{T_0^*}\right)_{cc}^{0.5} = \frac{w}{w^*}$$

$(T_0/T_0^*)_{cc}$ is the combined-change stagnation temperature ratio. Young [7] shows that assuming a linear variation of duct diameter with axial position, a relation between C_f and the change variables can be developed.

$$C_f = \frac{2fL}{D^*\left(\sqrt{\dfrac{A}{A^*}} - 1\right)}$$

The stagnation temperature ratio for Rayleigh flow may be taken from the simple change cases above and related to that for combined-change flows.

$$F'_{Ray} = F'_{cc}$$

$$\left(\frac{T_0}{T_0^*}\right)_{Ray} = \left(\frac{T_0}{T_0^*}\right)_{cc}^{1.5}$$

Therefore, a set of Rayleigh flow functions could be used to solve problems involving the combined-change process described above. For example, at a Mach number of 0.5, the following ratios are obtained from a table of Rayleigh flow functions with $k = 1.4$.

$$\frac{T_0}{T_0^*} = 0.69136,\ \frac{p}{p^*} = 1.7778 \text{ and } \frac{P_0}{P_0^*} = 1.1141$$

Then the value of the combined-change stagnation temperature ratio can be obtained from:

$$\left(\frac{T_0}{T_0^*}\right)_{Ray} = 0.69136 = \left(\frac{T_0}{T_0^*}\right)_{cc}^{1.5}$$

or

$$\left(\frac{T_0}{T_0^*}\right)_{cc} = 0.78187$$

from which the other variables may be determined as:

$$\left(\frac{A}{A^*}\right)_{cc} = (0.78187)^{0.25} = 0.94034$$

$$\left(\frac{w}{w^*}\right)_{cc} = (0.78187)^{0.5} = 0.88423$$

$$\frac{4fL_{max}}{D^*} = 2(-0.5)[(0.78187)^{0.125} - 1] = 0.030290$$

Using Eqs (7), (9) and (10) as checks with $M = 0.5$, $C_2/C_1 = 1$ and $k = 1.4$, the generalized function equivalents are listed below.

$$F' = 0.69136, \quad p' = 1.7778 \quad \text{and} \quad P_0' = 1.1141$$

While this insight is important and provides a logical transition from simple changes to combined changes, the primary benefit of the model described is to solve analytically combined-change problems without resorting to the types of artifices above. Figure 1 is a plot of Eqs (7) and (12) for a specific heat ratio, k, of 1.4 for a number of positive values of C_2/C_1. Negative C_2/C_1 values will be discussed later. Figure 1 shows that F' is zero when the Mach number is zero and increases to a maximum at a Mach number of one, after which it decreases with increasing Mach number. The plots corresponding to C_2/C_1 of 0 and 1 and $C_1 = 0$ are labelled generalized isentropic, Rayleigh and Fanno flows, respectively, since these plots represent combined-change as well as simple change flow as previously demonstrated.

Figures 2–4 are plots of T', p' and P_0' as functions of Mach number and C_2/C_1 for a specific heat ratio of 1.4. Figure 5 is a modified Mollier diagram for a number of positive values of C_2/C_1 and $k = 1.4$. This diagram shows that the behavior of combined-change flows, consistent with the flow model, is analogous to simple change flows for positive values of C_2/C_1. That is, the entropy of the flow increases as Mach one is approached from either the subsonic or supersonic side. The maximum entropy point at Mach one represents the same type of flow limitation that it did in simple change flow. If the change in independent variables is larger than that necessary to drive the flow to Mach one, a flow readjustment must take place.

NORMAL SHOCKS IN COMBINED-CHANGE FLOW

Since normal shocks take place in an axial length equal to a few mean free paths in the gas, a combined-change process involving a normal shock can be modelled without loss of generality as

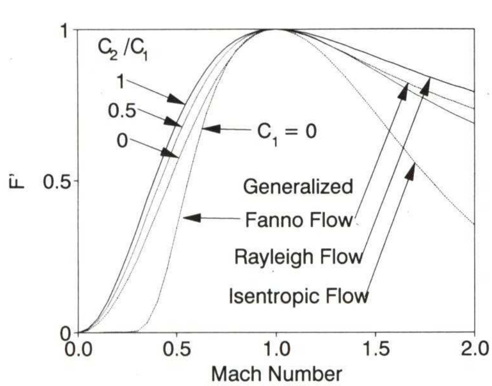

Fig. 1. F' vs Mach number for positive C_2/C_1 and $k = 1.4$.

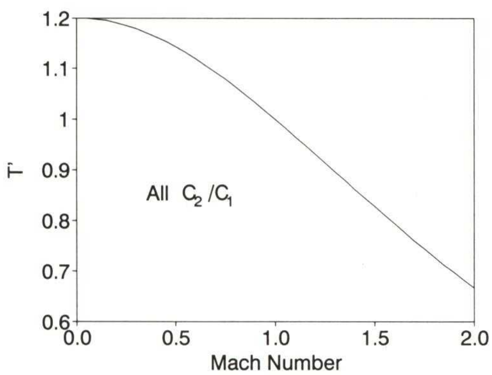

Fig. 2. T' vs Mach number for positive C_2/C_1 and $k = 1.4$.

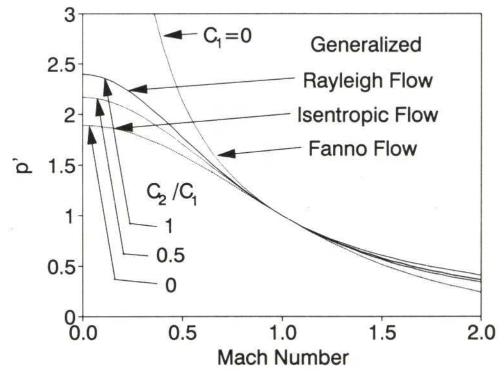

Fig. 3. p' vs Mach number for positive C_2/C_1 and $k = 1.4$.

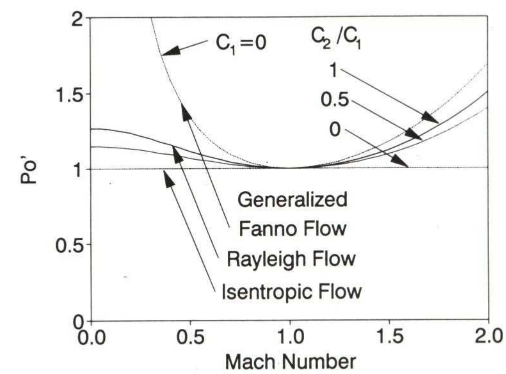

Fig. 4. P_0' vs Mach number for positive C_2/C_1 and $k = 1.4$.

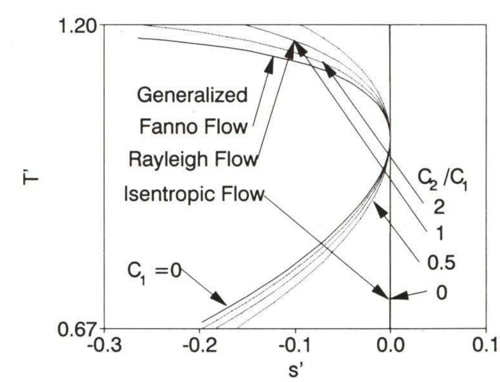

Fig. 5. Modified Mollier diagram for positive C_2/C_1 and $k = 1.4$.

a combined-change upstream of the shock, the shock and a continuation of the combined-change process downstream of the shock. An important consideration in examining the effects of normal shocks is the change induced in the independent variable at the Mach one location. For example, in isentropic flow the change in A* is used as a measure of the effect of normal shocks. For combined-change flows, the ratio of F^* across the shock will be developed as Fxy to indicate the effects of a normal shock as shown below.

$$Fxy = \left(\frac{F_x^*}{F_y^*}\right)^{C_1} = \left(\frac{F'_y}{F'_x}\right) \qquad (17a)$$

for all $C_2/C_1 \neq 0$ and

$$Fxy = \left(\frac{F_x^*}{F_y^*}\right)^{C_2} = \left(\frac{F'_y}{F'_x}\right) \qquad (17b)$$

for $C_1 = 0$. Figure 6 is a plot of Fxy as a function of supersonic Mach number for positive values of C_2/C_1. This figure shows that the only value of C_2/C_1 for which F^* does not change is one.

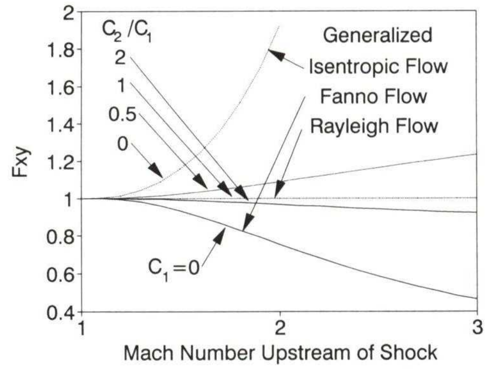

Fig. 6. Fxy from Eqs (17a–b) as a function of Mach number upstream of a normal shock.

CONSTANT PROPERTIES IN COMBINED-CHANGE FLOW

Constant properties will be obtained by taking the constitutive differential equation for that property (from Shapiro [1]) in terms of the independent variables and Mach number and equating the differential change in the parameter to zero.

Constant Mach number

For constant Mach number, dM^2 is set to zero.

$$\frac{(1-M^2)dM^2}{M^2\left[1+\frac{(k-1)}{2}M^2\right]} = -2\frac{dA}{A} + (1+kM^2)\frac{dT_0}{T_0} + kM^2\frac{4f dx}{D} + 2[1+(1-y)kM^2]\frac{dw}{w}$$

$$\frac{(1-M^2)dM^2}{M^2\left[1+\frac{(k-1)}{2}M^2\right]} = -2C_a\frac{dF}{F} + (1+kM^2)C_t\frac{dF}{F} + kM^2C_f\frac{dF}{F} + 2[1+(1-y)kM^2]C_w\frac{dF}{F}$$

$$0 = -2C_a + (1+kM^2)C_t + kM^2C_f + 2[1+(1-y)kM^2]C_w$$

Constant Mach number can be obtained from the equation above in two ways. The values of the change coefficients can be chosen at a particular Mach number to make this equation zero at that Mach number only. Those values of change coefficients are obtained from:

$$\frac{C_2}{C_1} = -\frac{1}{kM^2} \qquad (18)$$

or in terms of the change coefficients,

$$C_t(kM^2+1) + kM^2C_f + 2C_w[(1-y)kM^2+1] - 2C_a = 0$$

In order to obtain more specific results, at least one non-constant independent variable must be identified so that, in turn, the generalized variable F may be identified. Variable area cases will be considered first. In this case $C_a = 1$ and the following relationship for C_f is taken from Young [7].

$$C_f = \frac{2fL}{D_1\left(\frac{D_2}{D_1}-1\right)}$$

The change coefficient form of Eq. (18) may then be used to express the relation in terms of properties.

$$\left(\frac{\ln\frac{T_{02}}{T_{01}}}{\ln\frac{A_2}{A_1}}\right)(kM^2+1) + \frac{kM^2 2fL}{D_1\left(\frac{D_2}{D_1}-1\right)} + \left(\frac{\ln\frac{w_2}{w_1}}{\ln\frac{A_2}{A_1}}\right)2[(1-y)kM^2+1] - 2 = 0$$

Then Eq. (18) for constant Mach number and variable area becomes:

$$\left(\frac{T_{02}}{T_{01}}\right)^{(kM^2+1)}\left(\frac{A_2}{A_1}\right)^{\left[\frac{2fLkM^2}{D_1\left(\sqrt{\frac{A_2}{A_1}}-1\right)}-2\right]}\left(\frac{w_2}{w_1}\right)^{2[(1-y)kM^2+1]}=1 \qquad (18a)$$

from which the two variable cases are extracted and shown below.

Constant Mach number relations for variable area

Area and heat transfer

$$\frac{A_2}{A_1}=\left(\frac{T_{02}}{T_{01}}\right)^{\frac{kM^2+1}{2}}$$

Area and injection

$$\frac{A_2}{A_1}=\left(\frac{w_2}{w_1}\right)^{[(1-y)kM^2+1]}$$

Area and friction

$$D_2-D_1=fLkM^2$$

Zucrow and Hoffman [3] on pages 500–501 show the same result for the first and last cases above.

These results raise the interesting question, 'what would be the relationship for combined area change, friction, and heat transfer to obtain constant Mach number?' In this case, there would be an infinite number of solutions, since the resulting equations are not fully constrained. One class of solution from Eq. (18a) is:

$$\frac{A_2}{A_1}=\frac{T_{02}}{T_{01}}^{\frac{1+kM^2}{a}} \quad \text{and } D_2-D_1=\frac{kM^2fL}{b}$$

where

$$a+2b=2 \quad \text{and} \quad b\geqslant 0 \quad \text{for} \quad A_2>A_1$$

and

$$a+2b=2 \quad \text{and} \quad b\leqslant 0 \quad \text{for} \quad A_2<A_1.$$

The case of constant area will be examined since it changes the way in which the friction term is integrated from Eq. (3). Taking T_0 as variable and area as a constant, Eq. (18) becomes:

$$\left(\frac{T_{02}}{T_{01}}\right)^{(kM^2+1)}\exp\frac{4fLkM^2}{D}\left(\frac{w_2}{w_1}\right)^{2[(1-y)kM^2+1]}=1 \qquad (18b)$$

which is the same relation that would result if w were taken as the variable. The conditions for constant Mach number for each of the two variable cases involved are listed below.

Constant Mach number relations with constant area

Heat transfer and friction

$$\frac{T_{02}}{T_{01}}=\exp\frac{-4fLkM^2}{D(kM^2+1)}$$

Heat transfer and injection

$$\frac{T_{02}}{T_{01}}=\left(\frac{w_2}{w_1}\right)^{\frac{-2[(1-y)kM^2+1]}{(kM^2+1)}}$$

Injection and friction

$$\frac{w_2}{w_1}=\exp\frac{-2fLkM^2}{D[(1-y)kM^2+1]}$$

These results are useful in another context. If the Mach number goes through a stationary point, the location of that point can be determined from Eq. (18). For example, Beans [6] shows by numerical calculations that the sonic point in a converging–diverging nozzle is reached at a point downstream of the minimum area when friction is considered. Consider such a converging–diverging (C–D) nozzle constructed of a large number of truncated conical rings that closely approximate any desired nozzle contour. Further, consider this nozzle at the condition for which the back pressure is reduced just to the point that a Mach number of one is reached within the nozzle. At this point the Mach number goes through a maximum at $M=1$ and is therefore an inflection point. The conical ring which satisfies Eq. (18) for $M=1$ must contain constant Mach number flow at $M=1$. Using the area change and friction case of constant Mach number given previously, the slope of the nozzle wall is found at the sonic point at shown below by substituting $M=1$ and taking the limit as L approaches zero.

$$\frac{\mathrm{d}D}{\mathrm{d}x}=kf$$

This relationship agrees with the one Beans obtained through an analytic argument. While the argument above considered the case when the Mach number is maximum at $M=1$, the position of the sonic point should not move because the $M=1$ will remain a stationary point when the back pressure is reduced further.

A more general and perhaps straightforward approach is to use Eq. (18) at a Mach number of one and substitute into Eqs (1)–(6). As long as the independent variables are continuous functions of the axial dimension, x, and assuming that $A\alpha D^2$, the sonic point locations for cases involving area change can be found as shown below.

Variable area sonic point location

With heat transfer

$$\frac{dD}{dx} = \frac{1+k}{4}\frac{D}{T_0}\frac{dT_0}{dx}$$

With injection

$$\frac{dD}{dx} = \frac{D[1+k(1-y)]}{2w}\frac{dw}{dx}$$

With heat transfer and friction

$$\frac{dD}{dx} = kf + \frac{1+k}{4}\frac{D}{T_0}\frac{dT_0}{dx}$$

With injection and friction

$$\frac{dD}{dx} = \frac{D[1+k(1-y)]}{2w}\frac{dw}{dx} + kf$$

With heat transfer, injection and friction

$$\frac{dD}{dx} = kf + \frac{1+k}{4}\frac{D}{T_0}\frac{dT_0}{dx} + \frac{D[1+k(1-y)]}{2w}\frac{dw}{dx}$$

Beans did not present analytic expressions for the cases above.

In addition, Beans presents calculated cases in which the rate of change of area in a C–D nozzle is large just downstream of the minimum area and then becomes small near the nozzle exit. Beans shows that the supersonic Mach number increases at first, goes through a maximum and then decreases. The stationary point in this case is an inflection point. Beans's data for the friction factor, wall slope and area ratio as functions of x/L are used to calculate the inflection Mach number as a function of x/L. The point of intersection of the inflection Mach number and the Mach number calculated by Beans is at the maximum Mach number. The same procedure is used for the area change-heat transfer and the area change-heat transfer-friction cases presented by Beans. Figure 7 shows the data of Beans reproduced for these cases and the inflection Mach numbers calculated from Eq. (18). The intersection of the inflection Mach number with Beans's data occurs at the maximum as expected. The information gained from Eq. (18) can then be used as a tool in adjusting the variations in independent variables to achieve or eliminate points of inflection in the Mach number without necessarily calculating the complete Mach number distribution. For example, the steepness of the inflection Mach number curves in Fig. 7 indicates that there will be an inflection point even if the actual Mach number distribution is not known.

The second method of obtaining a constant Mach number is to choose the change coefficients so that each power of M is independently zero thereby giving a solution that is independent of Mach number.

$$-2C_a + C_t + 2C_w = 0 = C_1$$

$$C_t + C_f + 2(1-y)C_w = 0 = C_2$$

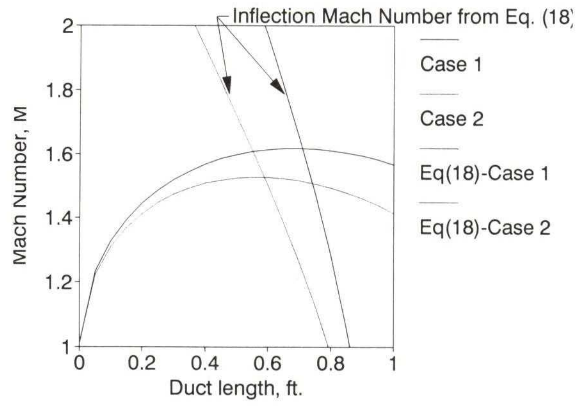

Fig. 7. Mach number in the diverging section of a converging–diverging nozzle after Beans [6].

An important result of the equations above is that anytime $C_1 = 0$ and $C_2 = 0$, the flow is at constant Mach number.

Constant velocity

For constant velocity, dV/V is set to zero.

$$\frac{dV}{V} = 0$$

$$-\frac{dA}{A} + \left[1 + \frac{(k-1)}{2}M^2\right]\frac{dT_0}{T_0} + \frac{k}{2}M^2\frac{4f dx}{D} + [1+(1-y)kM^2]\frac{dw}{w} = 0$$

$$-C_a\frac{dF}{F} + \left[1 + \frac{(k-1)}{2}M^2\right]C_t\frac{dF}{F} + \frac{k}{2}M^2C_f\frac{dF}{F} + [1+(1-y)kM^2]C_w\frac{dF}{F} = 0$$

$$-C_a + \left[1 + \frac{(k-1)}{2}M^2\right]C_t + \frac{k}{2}M^2C_f + [1+(1-y)kM^2]C_w = 0$$

The expression for any Mach number is then

$$-C_a + C_t + C_w = 0$$

$$\frac{k-1}{2k}C_t + \frac{1}{2}C_f + (1-y)C_w = 0$$

These latter two equations are equivalent to $C_2/C_1 = -1/k$ and $C_1 = -C_t$. Proceeding in a manner similar to that used for constant Mach number, the relations below are developed for constant velocity.

$$\left(\frac{T_{02}}{T_{01}}\right)^{\frac{(k-1)}{k}}\left(\frac{A_2}{A_1}\right)^{\frac{2fL}{D_1\left(\sqrt{\frac{A_2}{A_1}}-1\right)}}\left(\frac{w_2}{w_1}\right)^{2(1-y)} = 1$$

Constant velocity relations

Heat transfer, area and friction

$$\frac{T_{02}}{T_{01}} = \left(\frac{A_2}{A_1}\right)^{\overline{(k-1)D_1\left(\sqrt{\frac{A_2}{A_1}}-1\right)}^{-2fLk}}$$

Heat transfer, and injection

$$\frac{T_{02}}{T_{01}} = \left(\frac{w_2}{w_1}\right)^{\frac{-2k(1-y)}{(k-1)}}$$

Injection area and friction

$$\frac{A_2}{A_1} = \left(1 - \frac{fL}{(1-y)D_1}\right)^2 = \frac{w_2}{w_1}$$

An alternate solution is adiabatic and $C_1 = 0$ and $C_2 = 0$ which corresponds to constant Mach number. The relations corresponding to this solution are given below.

Constant velocity, adiabatic, variable area

With injection and friction

$$\frac{A_2}{A_1} = \left(1 - \frac{fL}{D_1(1-y)}\right)^2 = \frac{w_2}{w_1}$$

With injection

$$\frac{w_2}{w_1} = \frac{A_2}{A_1},\ y = 1$$

Constant speed of sound and temperature

For constant speed of sound, dc/c is set to zero.

$$\frac{dc}{c} = 0$$

$$M^2\frac{dA}{A} + \frac{(1-kM^2)}{(k-1)}\left[1 + \frac{(k-1)}{2}M^2\right]\frac{dT_0}{T_0} - \frac{k}{2}M^4\frac{4f dx}{D} - M^2[1 + k(1-y)M^2]\frac{dw}{w} = 0$$

$$\left\{M^2C_a + \frac{(1-kM^2)}{(k-1)}\left[1 + \frac{(k-1)}{2}M^2\right]C_t - \frac{k}{2}M^4C_f - M^2[1 + k(1-y)M^2]C_w\right\}\frac{dF}{F} = 0$$

$$M^2C_a + \frac{(1-kM^2)}{(k-1)}\left[1 + \frac{(k-1)}{2}M^2\right]C_t - \frac{k}{2}M^4C_f - M^2[1 + k(1-y)M^2]C_w = 0$$

The expression for any Mach number is shown below.

$$C_t = 0$$

$$C_a - C_w = 0 \quad \text{or} \quad C_a = C_w$$

$$-\tfrac{1}{2}C_f - (1-y)C_w = 0 \quad \text{or} \quad C_f = -2(1-y)C_a$$

These latter two equations are equivalent to $C_1 = 0$ and $C_2 = 0$. The process is therefore also at constant Mach number. Relations for this case are the same as those for the same case in constant velocity flow. It is shown later that for $C_1 = 0$ and $C_2 = 0$, static and stagnation pressures are also constant. Since the speed of sound is proportional to the square root of temperature, the case above for constant c applies for constant T. This case may be reduced to $C_a = 1 = C_w$ and $C_f = -2(1-y)$. If the velocity ratio is less than one, the area ratio must be less than one to maintain a positive friction coefficient. The result then would appear to be counter-intuitive. The process involves area change, friction and mass extraction with no change in Mach number, temperature, pressure or stagnation pressure. Actually, the mass extraction process with a velocity ratio less than unity is removing lower momentum flow at exactly the rate at which friction is decreasing momentum. While this is a theoretical possibility, two-dimensional effects would make realization difficult.

Constant density

For constant density, $d\rho/\rho$ is set to zero.

$$\frac{d\rho}{\rho} = 0$$

$$M^2\frac{dA}{A} - \left[1 + \frac{(k-1)}{2}M^2\right]\frac{dT_0}{T_0} - kM^2\frac{4f dx}{D} - [1 + (1-y)]kM^2\frac{dw}{w} = 0$$

$$M^2C_a\frac{dF}{F} - \left[1 + \frac{(k-1)}{2}M^2\right]C_t\frac{dF}{F} - kM^2C_f\frac{dF}{F} - [1 + (1-y)k]M^2C_w\frac{dF}{F} = 0$$

$$M^2C_a - \left[1 + \frac{(k-1)}{2}M^2\right]C_t - \frac{k}{2}M^2C_f - [1 + (1-y)k]M^2C_w = 0$$

The expression for any Mach number is shown below.

$$C_t = 0$$

$$C_a - \frac{k}{2}C_f - [(1-y)k + 1]C_w = 0$$

These latter two equations are equivalent to $C_2/C_1 = -1/k$ for adiabatic flow. Specific independent variable relations are obtained for this case as shown.

Constant density relations

With injection, area and friction

$$\left(\frac{w_2}{w_1}\right)^{[1+k(1-y)]} = \left(\frac{A_2}{A_1}\right)^{\left[1-\frac{fLk}{D_1\left(\sqrt{\frac{A_2}{A_1}}-1\right)}\right]}$$

Area and friction

$$\frac{A_2}{A_1} = \left(1 + \frac{fLk}{D_1}\right)^2$$

Area and injection

$$\left(\frac{w_2}{w_1}\right)^{[1+k(1-y)]} = \frac{A_2}{A_1}$$

Friction and injection

$$\frac{w_2}{w_1} = \exp\frac{-2fLk}{D[1+k(1-y)]}$$

An alternate solution is $C_1 = 0$ and $C_2 = 0$ for adiabatic flow which is the same as one case of constant velocity flow.

Constant pressure

For constant static pressure, dp/p is set to zero.

$$\frac{dp}{p} = 0$$

$$\frac{dA}{A} - \left[1 + \frac{(k-1)}{2}M^2\right]\frac{dT_0}{T_0} - [1+(k-1)M^2]\frac{4f dx}{D}$$

$$+ [-2 + y - (1-y)(k-1)M^2]\frac{dw}{w} = 0$$

$$C_a\frac{dF}{F} - \left[1 + \frac{(k-1)}{2}M^2\right]C_t\frac{dF}{F}$$

$$- [1+(k-1)M^2]C_f\frac{dF}{F}$$

$$- [2 - y + (1-y)(k-1)M^2]C_w\frac{dF}{F} = 0$$

$$C_a - \left[1 + \frac{(k-1)}{2}M^2\right]C_t - \frac{[1+(k-1)M^2]}{2}C_f$$

$$+ [-2 + y - (1-y)(k-1)M^2]C_w = 0$$

The expression for any Mach number is shown below.

$$C_a - C_t - \tfrac{1}{2}C_f - (2-y)C_w = 0$$

$$C_t + C_f - 2(1-y)C_w = 0$$

These latter two equations are equivalent to $C_1 = 0$ and $C_2 = 0$. The process is therefore also at constant Mach number.

Constant stagnation pressure

For constant stagnation pressure, dP_0/P_0 is set to zero.

$$\frac{dP_0}{P_0} = 0$$

$$-\frac{1}{2}\frac{dT_0}{T_0} - \frac{1}{2}\frac{4f dx}{D} - (1-y)\frac{dw}{w} = 0$$

$$-\frac{1}{2}C_t\frac{dF}{F} - \frac{1}{2}C_f\frac{dF}{F} - (1-y)C_w\frac{dF}{F} = 0$$

$$C_t + C_f + 2(1-y)C_w = 0$$

$$\text{any } C_a$$

These change coefficients are converted into relations among the independent variable as shown previously.

$$\left(\frac{T_{02}}{T_{01}}\right)\left(\frac{A_2}{A_1}\right)^{\frac{2fL}{D_1\left(\sqrt{\frac{A_2}{A_1}}-1\right)}}\left(\frac{w_2}{w_1}\right)^{2(1-y)} = 1$$

Constant stagnation pressure relations

Heat transfer and injection

$$\frac{T_{02}}{T_{01}} = \left(\frac{w_2}{w_1}\right)^{-2(1-y)}$$

Friction and injection

$$\frac{4fL}{D} = -2(1-y)$$

Injection

$$y = 1$$

Area, injection and friction

$$\frac{w_2}{w_1} = \left(\frac{A_2}{A_1}\right)^{\frac{-fL}{(1-y)D_1\left(\sqrt{\frac{A_2}{A_1}}-1\right)}}$$

Area, heat transfer and friction

$$\frac{T_{02}}{T_{01}} = \left(\frac{A_2}{A_1}\right)^{\frac{-2fL}{D_1\left(\sqrt{\frac{A_2}{A_1}}-1\right)}}$$

Constant entropy

For constant entropy, ds/c_v is set to zero.

$$\frac{ds}{c_v} = 0$$

$$\left[1 + \frac{(k-1)}{2}M^2\right]\frac{dT_0}{T_0} + \frac{(k-1)}{2}M^2\frac{4f dx}{D}$$

$$+ (1-y)(k-1)M^2\frac{dw}{w} = 0$$

$$\left[1+\frac{(k-1)}{2}M^2\right]C_t\frac{\mathrm{d}F}{F}+\frac{(k-1)}{2}M^2C_f\frac{\mathrm{d}F}{F}$$
$$+(1-y)(k-1)M^2C_w\frac{\mathrm{d}F}{F}$$
$$=0$$
$$C_t=0$$
$$C_t+C_f+2(1-y)C_w=0=C_2$$

These latter two equations are equivalent to $C_2/C_1=0$ for an adiabatic process. The three specific cases corresponding to these isentropic conditions are shown below.

Constant entropy relations

Area, friction and injection

$$\frac{w_2}{w_1}=\left(\frac{A_2}{A_1}\right)^{\dfrac{-fL}{D_1(1-y)\left(\sqrt{\frac{A_2}{A_1}}-1\right)}}$$

Friction and injection

$$\frac{w_2}{w_1}=\exp\frac{-2fL}{D(1-y)}$$

Area and injection

$y=1$ arbitrary A, w

Discussion of some constant-property results

A summary of all constant-property conditions is listed in Table 2. These constant-property conditions are based on the relationships between independent variables assumed for the generalized flow model. There are many other possible relationships between the independent variables in addition to those used. The conditions obtained with this model are, therefore, not complete. An example of a case not included is that of isothermal flow with simple friction. The variation of T_0 and $4fL/D$ is not of the form of Eqs (2)–(3) for this case.

The conditions are correct within the bounds of the model limitations. There are a few cautions. The model permits negative friction coefficients, extracting mass at velocity ratios greater than one, and other physically unattainable conditions. The model does not consider real effects such as multi-dimensional flow or shocks.

Table 2. Summary of constant-property conditions

Property	Conditions necessary to hold constants
M	$C_2/C_1=-1/kM^2$ or $C_1=0$ and $C_2=0$
V	$C_1=-C_t$ and $C_2/C_1=-1/k$ or adiabatic and $C_1=0$ and $C_2=0$
c, T	adiabatic and $C_1=0$ and $C_2=0$
p	$C_1=0$ and $C_2=0$
ρ	adiabatic and $C_2/C_1=-1/k$ or adiabatic and $C_1=0$ and $C_2=0$
P_0	$C_2/C_1=0$ any C_1 or $C_1=0$ and $C_2=0$
s	adiabatic, $C_2/C_1=0$ any C_1

The case of $C_1=0$ and $C_2=0$ results in constant Mach number, speed of sound, static temperature and pressure, and stagnation pressure. With the Mach number determined to be constant and $C_1=0$, these results can be obtained from the mass flow functions as shown below

$$\frac{w_2\sqrt{T_{01}}}{p_1A_1}=g(M_1,k)$$

$$\frac{w_1\sqrt{T_{01}}}{p_1A_1}\frac{p_1}{p_2}\frac{A_1}{A_2}\sqrt{\frac{T_{02}}{T_{01}}}\frac{w_2}{w_1}=\frac{w_1\sqrt{T_{02}}}{p_2A_2}=g(M_2,k)$$

but $M_2=M_1$, therefore

$$g(M_1,k)\frac{p_1}{p_2}\left(\frac{F_2}{F_1}\right)^{-C_a}\left(\frac{F_2}{F_1}\right)^{\frac{C_t}{2}}\left(\frac{F_2}{F_1}\right)^{C_w}=g(M_1,k)$$

$$\frac{p_1}{p_2}\left(\frac{F_2}{F_1}\right)^{-C_a+\frac{C_t}{2}+C_w}=1.$$

The exponent of F_2/F_1 above is $-C_1/2$, or zero. The value of p_1/p_2 is then 1, implying constant pressure. A similar result is obtained using the stagnation pressure mass flow function, with the result that P_0 is constant.

An interesting consequence of both C_1 and C_2 being zero is that for $y=0$, $C_f=-2C_a$ and $C_t+2C_w=2C_a$. If this constant Mach number process involves area change, it involves friction. The negative sign in the relation between friction and area-change coefficients permits decreasing area only, so that the friction coefficient remains positive.

The cases for which $C_2/C_1=-1/k$ involve a set of generalized flow functions which are calculated from Eqs (7)–(11) and are included as Table 3. Special variables are defined for this table to demonstrate constant velocity when $C_1=-C_t$ and constant ρ when the flow is adiabatic. These variables are shown below.

$$V'=\frac{V}{V^*}\text{ for }C_1=-C_t$$

An expression for the stagnation temperature ratio for this case is shown below.

$$\frac{T_0}{T_0^*}=\left(\frac{F}{F^*}\right)^{C_t}=\left(\frac{F}{F^*}\right)^{-C_1}=\frac{1}{F'}$$

The temperature ratio may then be found in terms of F'.

$$\frac{T}{T^*}=\frac{T'}{F'}$$

Finally, the velocity may be expressed in terms of the Mach number and the speed of sound. The

Table 3. Generalized flow functions for $C_2/C_1 = -1/k$ and $k = 1.4$

M	F'	P_0'	p'	T'	s'	V'	ρ'
0.0	0.000000	0.633938	1.200000	1.200000	0.182322	1	1
0.1	0.011076	0.637112	1.197605	1.197605	0.180324	1	1
0.2	0.047619	0.646693	1.190476	1.190476	0.174353	1	1
0.3	0.106090	0.662852	1.178782	1.178782	0.164482	1	1
0.4	0.186047	0.685877	1.162791	1.162791	0.150823	1	1
0.5	0.285714	0.716176	1.142857	1.142857	0.133532	1	1
0.6	0.402985	0.754282	1.119403	1.119403	0.112796	1	1
0.7	0.535519	0.800853	1.092896	1.092896	0.088831	1	1
0.8	0.680851	0.856682	1.063830	1.063830	0.061875	1	1
0.9	0.836489	0.922703	1.032702	1.032702	0.032179	1	1
1.0	1.000000	1.000000	1.000000	1.000000	0.000000	1	1
1.1	1.169082	1.089810	0.966184	0.966184	−0.03440	1	1
1.2	1.341614	1.193538	0.931677	0.931677	−0.07077	1	1
1.3	1.515695	1.312765	0.896861	0.896861	−0.10885	1	1
1.4	1.689655	1.449255	0.862069	0.862069	−0.14842	1	1
1.5	1.862069	1.604970	0.827586	0.827586	−0.18924	1	1
1.6	2.031746	1.782076	0.793651	0.793651	−0.23111	1	1
1.7	2.197719	1.982962	0.760456	0.760456	−0.27384	1	1
1.8	2.359223	2.210242	0.728155	0.728155	−0.31724	1	1
1.9	2.515680	2.466776	0.696864	0.696864	−0.36116	1	1
2.0	2.666667	2.755676	0.666667	0.666667	−0.40547	1	1

speed of sound is taken as the square root of temperature.

$$V' = M\sqrt{\frac{T}{T^*}} = M\sqrt{\frac{T'}{F'}}$$

For adiabatic flow, T_0/T_0^* is one, so the density may be found from the equation of state as p'/T'.

$$\rho' = \frac{\rho}{\rho^*}$$

$$\rho' = \frac{p'}{T'} \text{ for } C_t = 0$$

By design, the definitions of both V' and ρ' should be one at all values of Mach number to demonstrate that they are constant.

The last cases of constant properties involve stagnation pressure and entropy. In the latter case the flow must also be adiabatic. Table 4 contains the generalized flow functions for $C_2/C_1 = 0$. Constant stagnation pressure is clearly shown by Table 4. While s' is also shown to be zero, the definition of s' given above Eq. (11) should be closely examined. Entropy is zero when s' is zero and $C_t = 0$. Therefore, in addition to the conditions for Table 4, the process must also be adiabatic. The constant parameter conditions $C_2/C_1 = -1/k$ and $C_2/C_1 = -1/kM^2$ merit further examination. Figure 8 is the $T' - s'$ diagram (a modified Mollier diagram) for values of $C_2/C_1 = -1$, $-1/k$, and -0.5. Figures similar to Fig. 8 were first observed in cases of mass injection with $y > 1$ (Hodge and Young [8]). These figures indicate that the constant Mach number associated with $C_2/C_1 = -1/kM^2$ is a singularity in a more general flow process. Constant velocity and density are associated with the $C_2/C_1 = -1/k$ locus which divides flow behavior into two distinct regimes. One regime is characterized by subsonic singularity Mach num-

Table 4. Generalized flow functions for $C_2/C_1 = 0$ and $k = 1.4$

M	F'	P_0'	p'	T'	s'
0.0000	0.0000	1.0000	1.8929	1.2000	0.0000
0.1000	0.0295	1.0000	1.8797	1.1976	0.0000
0.2000	0.1139	1.0000	1.8409	1.1905	0.0000
0.3000	0.2415	1.0000	1.7784	1.1788	0.0000
0.4000	0.3955	1.0000	1.6953	1.1628	0.0000
0.5000	0.5570	1.0000	1.5958	1.1429	0.0000
0.6000	0.7083	1.0000	1.4841	1.1194	0.0000
0.7000	0.8350	1.0000	1.3647	1.0929	0.0000
0.8000	0.9277	1.0000	1.2418	1.0638	0.0000
0.9000	0.9825	1.0000	1.1190	1.0327	0.0000
1.0000	1.0000	1.0000	1.0000	1.0000	0.0000
1.1000	0.9843	1.0000	0.8866	0.9662	0.0000
1.2000	0.9418	1.0000	0.7806	0.9317	0.0000
1.3000	0.8795	1.0000	0.6832	0.8969	0.0000
1.4000	0.8045	1.0000	0.5948	0.8621	0.0000
1.5000	0.7229	1.0000	0.5156	0.8276	0.0000
1.6000	0.6398	1.0000	0.4454	0.7937	0.0000
1.7000	0.5589	1.0000	0.3835	0.7605	0.0000
1.8000	0.4829	1.0000	0.3294	0.7282	0.0000
1.9000	0.4134	1.0000	0.2825	0.6969	0.0000
2.0000	0.3512	1.0000	0.2419	0.6667	0.0000

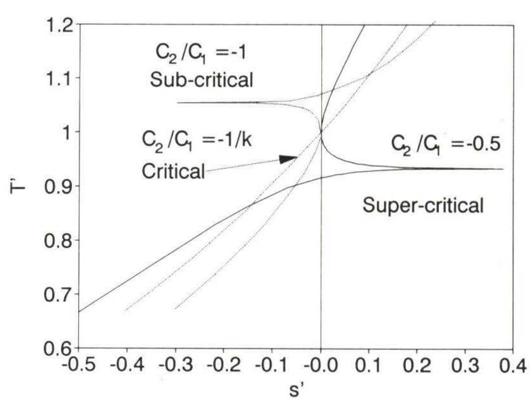

Fig. 8. $T - s$ diagram for $C_2/C_1 = -1, -1/k$, and -0.5 and $k = 1.4$.

bers which will be called the subcritical regime. The other regime is characterized by supersonic Mach numbers and will be called supercritical.

Figure 9 shows the generalized independent variable function, F', from Eq. (5), as a function of Mach number. Increasing F' drives subsonic flow processes toward the subcritical singularity while decreasing F' drives supersonic flow processes toward the supercritical singularity. Figure 10 shows P_0' from Eq. (10) as a function of Mach number. P_0' increases as a subcritical singularity is approached and decreases as a supercritical singularity is approached. For subcritical cases, there is a local minimum in P_0', and for supercritical cases a local maximum at $M = 1$. p' from Eq. (9) is shown as a function of Mach number in Fig. 11. The behavior of p' is similar to that of P_0' except for the inflection point. The inflection point in p' is at a Mach number of

$$M = \sqrt{\frac{-\left(\frac{C_2}{C_1}+1\right)}{\frac{C_2}{C_1}(k-1)}}$$

and is a local minimum for subcritical cases as long as $C_2/C_1 > -1$ and a local maximum for supercritical cases. For $C_2/C_1 < -1$ there is no inflection point.

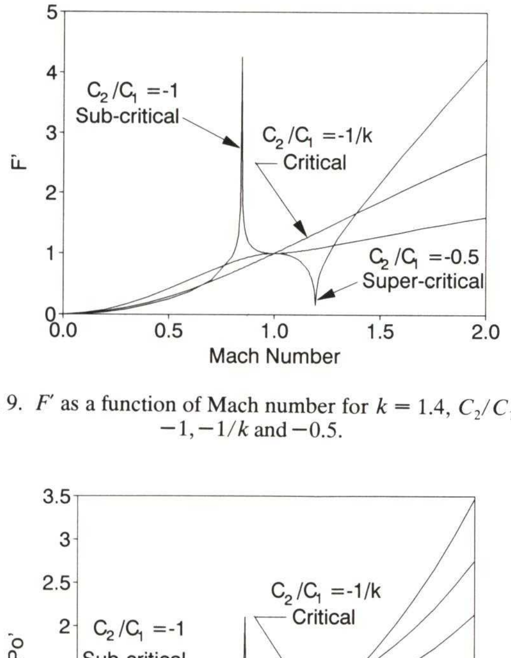

Fig. 9. F' as a function of Mach number for $k = 1.4$, $C_2/C_1 = -1, -1/k$ and -0.5.

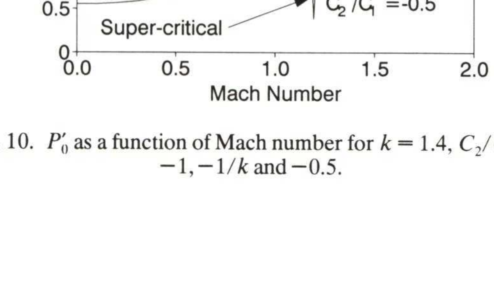

Fig. 10. P_0' as a function of Mach number for $k = 1.4$, $C_2/C_1 = -1, -1/k$ and -0.5.

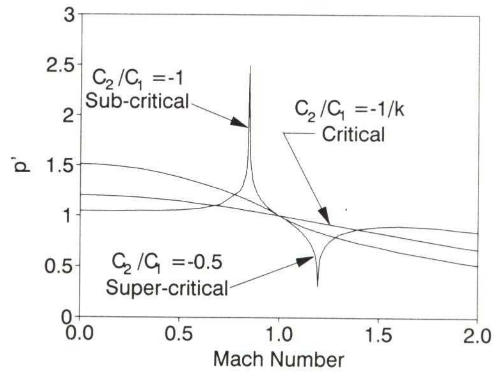

Fig. 11. P' as a function of Mach number for $k = 1.4$, $C_2/C_1 = -1, -1/k$ and -0.5.

A NUMERICAL EXAMPLE

Pratt and Heiser [9], in analyzing a dual-mode scramjet engine in the ramjet mode, have modelled the burner section as an area and stagnation temperature change of a one-dimensional flow of a perfect gas. Their analysis technique starts with the Mach number entering the burner section and then uses an assumed stagnation temperature distribution with axial dimension to numerically integrate the constitutive differential equation and hence to calculate the properties downstream. This process must be carried out iteratively since the 'thermal throat' ($M = 1$) must just be reached at the end of the axial length. The method presented here has three advantages. The method is analytic and thereby the number of calculations is reduced, especially for parametric studies. Also, the properties at the 'thermal throat' are given explicitly, thereby eliminating the need for iteration. The third advantage lies in the assumption that there is a relationship between the stagnation temperature and area. This permits a solution independent of how the stagnation temperature varies with axial dimension. As an example, take the Mach number at the burner entrance as 0.3. For this case, the generalized variable F will be taken as T_0, therefore, the relation between T_0 and A is given by,

$$\frac{A}{A_1} = \left(\frac{T_0}{T_{01}}\right)^{C_a}$$

where the 1 state is that entering the burner section and the c state is at the 'thermal throat'. When the axial stagnation temperature distribution is determined, the area variation follows as

$$\frac{T_0}{T_{01}} = f(x)$$

then

$$\frac{A}{A_1} = f(x)^{C_a}.$$

From Eqs (5)–(6) the values of C_1 and C_2 may be determined and then the ratio C_2/C_1.

$$\frac{C_2}{C_1} = \frac{1}{(1-2C_a)}$$

A number of values of C_a will be used to illustrate positive, zero and negative values of C_2/C_1. The stagnation temperature, area and stagnation pressure ratios across the burner section are shown in Table 5 as calculated from Eqs (7), (10), (12) and (14) as appropriate for the value of C_2/C_1.

The first entry for $C_a = 0$ corresponds to generalized Rayleigh flow and in this case it is actually simple Rayleigh flow. The entry for $C_a = 0.5$ corresponds to generalized Fanno flow even though there is no friction. The entry for $C_a = 0.563$ corresponds to Eq. (18) for a constant Mach number of 0.3. This latter value then describes an upper limit on the rate of area increase with stagnation temperature increase. If the limit is exceeded, the flow will decelerate instead of accelerating through the sonic point.

Table 5. Burner ratios

C_a	C_2/C_1	T_{0c}/T_{01}	A_c/A_1	P_{0c}/P_{01}
0.000	1.00	2.883	1	0.834
0.250	2.00	5.598	1.538	0.756
0.500	∞	200.2	14.15	0.491
0.563	−7.94	∞	∞	0

CONCLUSIONS

A combined change one-dimensional compressible flow model developed by Young [7] is shown to provide a logical extension of students' understanding of simple change flows to include combined-change flows. The first step in this extension is to show that each of the simple change processes with which the students are already familiar, actually represent an infinite number of combined-change processes which have the same value of C_2/C_1. The next step in the extension is to show that simple change processes are part of a spectrum of positive valued C_2/C_1 combined-change processes which exhibit behavior similar to that of simple change processes.

The behaviour of combined-change flow for negative values of C_2/C_1 is approached by using the model for constant property analysis. Properties analyzed include Mach number, velocity, speed of sound, density, static pressure and temperature, stagnation pressure and entropy. Two cases are found for constant Mach number. One case, $C_1 = C_2 = 0$, also involves constant pressure and stagnation pressure. If in addition the process is adiabatic, speed of sound, density, velocity, entropy and static temperature are constant. The other case, $C_2/C_1 = -1/kM^2$, is shown to coincide with the Mollier diagram singularities first observed by Hodge and Young [8] for cases of simple mass injection with $y > 1$. These singularities exhibit distinctly different behavior for subsonic and supersonic Mach numbers. These types of behavior are separated by the $C_2/C_1 = -1/k$ locus on a $T' - s'$ diagram. The $C_2/C_1 = -1/k$ locus is shown to represent a constant-density process when the flow is adiabatic, and a constant-velocity process when $C_t = -C_1$. This case of constant Mach number is also shown to describe the location of the sonic point downstream of the minimum area of a converging–diverging nozzle. This case also assists in determining whether a Mach number inflection point will exist in a given flow.

The model presented by Young [7] defines the thermodynamic path of the combined-change process and therefore, the model is not completely general. Despite this lack of generality, the model is shown to be sufficiently robust to provide new insight into one-dimensional compressible flow.

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