Int. J. Engng Ed. Vol. 9, No. 4, pp. 305–321, 1993
Printed in Great Britain.

0742–0269/91 $3.00+0.00

Turbo-compressors and Turbines as Conjugate Machines and a Compressor as a Machine with a Non-zero Lower Limit to its Isentropic Efficiency*

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The paper discusses the differences and similarities between compressors and turbines. After an explanation that treating turbomachines as steady flow devices is an approximation, it is pointed out that isentropic and polytropic efficiencies are simply work ratios. It is also shown that there is a non-zero lower limit to the isentropic efficiency of a compressor. The objectives of the paper are to provide further understanding of the processes and interactions associated with turbomachinery and, on this basis, develop a series of unified equations which may be applied to compressors and turbines. These ideas can be included in final-year degree courses in power plants and aircraft propulsion. The correlation of pressure ratios and temperature ratios across turbomachines, using a parameter of the form Pv^n, *is examined and limits are established. Attempts to operate a compressor below the non-zero lower limit of isentropic efficiency are shown to result in its being changed into a non-compression pumping machine. It is concluded that the unified equations which have been developed will be helpful to students and useful for computer modellers developing algorithms for simulating compressors and turbines.*

1. The paper discusses material for a course in: Thermodynamics and turbo-machinery.
2. Students of the following engineering departments are appropriate for the course: Mechanical, aeronautical, chemical, marine.
3. Level of the course: Final year degree.
4. Mode of presentation: Lectures, practicals and some software packages.
5. Is the material presented in a regular or in an elective course: Elective course.
6. Class hours required to cover the materials: 40 hours total, say 2 hours on new material.
7. Student homework and revision hours required for the materials: Homework and revision should match number of lecture hours.
8. Description of the novel aspects presented in the paper: Novel aspects are conjugate turbo-machines and a non-zero efficiency for the compressor.
9. The standard text recommended for the course, in addition to author's notes: Cohen, Rogers and Saravanamuttoo, *Gas Turbine Theory* (ref. [5] in paper).
10. The material is not covered in the text. The discussion in the text is different in the following aspects: The material is new, it is not covered in the text
11. Other comments: The paper should make students and teachers think, it should appeal particularly to intuitors and global learners.

NOMENCLATURE

Capital letters

C	specific heat capacity used with subscripts $_p$ and $_v$
P	pressure
R	characteristic gas constant
T	temperature
$\dot{W}_x$	power
X, Y, Z	cartesian coordinates used in Appendices 1 and 2

Lower-case letters

a	modulus of the complex number used in Appendix 2
c	constant in straight-line equation in Appendix 1
h	specific enthalpy
m	gradient in straight-line equation in Appendix 1
$\dot{m}$	mass flow rate
n	polytropic exponent

* Accepted 12 March 1993

n^1	exponent corresponding to non-compression pumping machine
q	heat transfer for unit mass
s	specific entropy
u	specific internal energy
v	specific volume
w	work for unit mass
$x =$	$(P_2/P_1)/(\gamma - 1)/\gamma$ isentropic temperature ratio for compressor
$y =$	$(P_3/P_4)/(\gamma - 1)/\gamma$ isentropic temperature ratio for turbine

Greek letters

$\gamma = C_p/C_v$	isentropic exponent
η	isentropic efficiency used with subscripts $_c$ and $_t$
η_p	polytropic efficiency used with subscripts $_c$ and $_t$
ξ	work ratio parameter, used with subscripts $_{p, c, m}$ and $_t$
$\theta =$	T_3/T_1 temperature ratio, used in Appendix 2
Φ	angle in Argand diagram used in Appendix 2
ρ	density of gas.

Subscripts

1	at inlet to the compressor
2	at outlet from compressor
3	at inlet to turbine
4	at outlet from turbine
a	actual conditions
c	compressor
i	isentropic conditions
in	at inlet conditions
min	minimum value
out	at outlet conditions
p	polytropic and also at constant pressure
ppm	polytropic for non-compression pumping machine
t	turbine
v	at constant volume

Special symbol

○	Circle around the symbol denotes a reference state used in Appendices 1 and 2.

FOREWORD—THE EDUCATIONAL ASPECTS

SPARKES [1] has put forward a framework of learning which makes a distinction between knowledge and understanding. Felder and Silverman [2] have described different types of learners. They explain that intuitors are the people who prefer principles and theories that are good at grasping new concepts. Sensors, on the other hand, like facts, data and experiments; they like solving problems by standard methods and do not like complications. Sensors are good at memorizing facts. Another dimension Felder and Silverman discuss concerns sequential and global learners. Sequential learners are comfortable with a logically ordered progression of material and they follow and understand as the material is presented. They follow a linear reasoning process when solving problems.

From a technical viewpoint, the material presented in this paper is relevant to final-year degree courses in thermal power or, perhaps, more specialized courses in turbo-machinery and aircraft propulsion. Educationally, the ideas presented may appeal to those who learn intuitively and globally. Generally, lecturers are intuitors but most students are sensors. The significance of the conjugate relationships, unified equations and the non-zero lower limit of the compressor isentropic efficiency discussed in the paper may have to be carefully explained to sensors and sequential learners, but it should be attractive to intuitors and global learners. The material given in the paper is, therefore, directed at lecturers delivering courses similar to those mentioned. The objectives of the paper are to provide further understanding of the overall thermo-fluid processes that occur in compressors and turbines.

The goal of the paper is to encourage lecturers to make small but important changes in the presentation of their courses.

INTRODUCTION

Turbo-compressors and turbines may be regarded as conjugate or reciprocal machines. The compressor is a work-consuming, pressure and temperature increasing device, whereas a turbine is a work-producing, pressure and temperature reducing device, but they are both entropy producing machines.

The word conjugate is preferred in this case on account of the similarity between the diagrams shown in Figs 1 and 2 and that of an Argand diagram showing a complex conjugate pair. It is convenient to defer further discussion of this topic until other aspects, given in the body of the paper and in Appendix 1, have been introduced. The analysis relevant to the conjugate nature of the

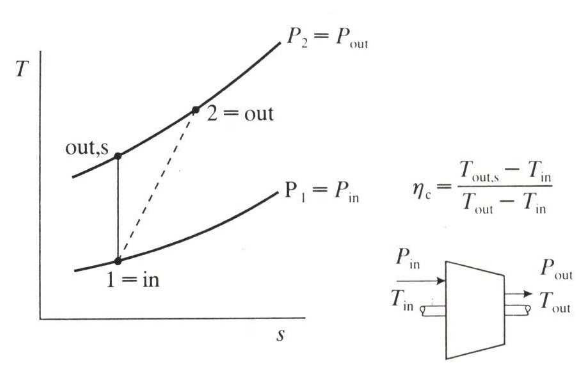

Fig. 1. Temperature–entropy diagram for a compressor.

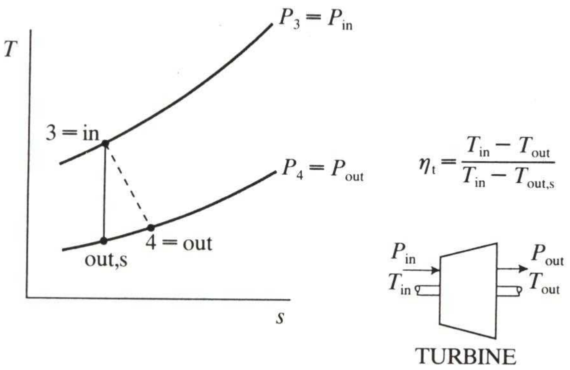

Fig. 2. Temperature–entropy diagram for a turbine.

compressor and turbine in given in Appendix 2. Turbo-machines are almost always regarded as steady flow machines but, strictly, this is not the case. Turbo-machines may be regarded as machines which operate overall on a steady inflow and steady outflow but the work interactions take place during internal, unsteady flows. Although this topic is not the subject of the present paper, it is relevant to mention that it has been studied by Dean [3] and Preston [4]. It is also appropriate to mention now that: (i) if a reversible flow has a work interaction, then the flow must be unsteady; and (ii) if a steady flow produces a work interaction, then it must be an irreversible process.

The processes considered in this paper are regarded as steady inflow and steady outflow but, as the through flows considered may be reversible or irreversible with work interactions, the internal flows are unsteady as they interact with individual blades. The machines considered in this paper are restricted to compressors and gas turbines and we simplify the analysis by treating the working fluid as a perfect gas.

Returning to the reciprocal nature of the compressor and the turbine, one object of this paper is to seek reciprocal relationships for the compressor and turbine and, accordingly, develop unified equations. In the course of doing this, a non-zero value was discovered for the minimum isentropic efficiency and the polytropic efficiency of the compressor. The minimum polytropic efficiency of a compressor is

$$\eta_{pc} = \frac{\gamma - 1}{\gamma}$$

and a further objective of the paper is to explain this.

In what follows, the basic ideas involved in applying the steady flow energy equation to the inflow and outflow of a control volume containing a turbo-machine are discussed. However, the details of the internal flows in the control volume and, in particular, the internal flows in the individual blade passages of compressors and turbines are not discussed. The limits of the isentropic efficiencies mentioned and expressions for the minimum polytropic efficiency and isentropic efficiency of the compressor are derived.

Next, the unified equations for the turbo-machine and the transformations needed to apply them are discussed. The results of limits of the efficiencies and the unified equations are considered; the main conclusions drawn are that there is a non-zero minimum compressor efficiency, which in the polytropic form is

$$\eta_{pc,min} = \frac{\gamma - 1}{\gamma}$$

and a set of unified turbo-machinery equations have been derived together with the required transformations needed to apply them to compressors and turbines.

BASIC IDEAS

The thermodynamic model selected to represent the turbo-machine is the steady flow energy equation, with a perfect gas flowing through a control volume [5]. The turbo-machine is assumed to be adiabatic and the change of elevation is regarded as negligible. The steady flow energy equation is written in terms of the stagnation states as

$$\dot{W}_x = \dot{m}\,(h_{out} - h_{in}) \tag{1}$$

here the notation used for the work interaction is that transfer to the control volume is the positive. This modern rational notation was recently explained by Mayhew [6]. In this paper, pressures and temperatures are stagnation state values unless otherwise stated.

For the perfect gas, equation (1) is,

$$\dot{W}_x = \dot{m}C_p(T_{out} - T_{in}) \tag{2}$$

Equations (1) and (2) are applicable to both a compressor and a turbine. Logically, for a compressor $T_{out} > T_{in}$ and the power term is positive. Corresponding to the gas turbine, $T_{out} < T_{in}$ and the power term represents a negative quantity.

The application of the Second Law of Thermodynamics to the turbo-machine [7] gives $s_{out} > s_{in}$ and, as this applies both to the compressor and the turbine, it leads to the basis of the isentropic efficiency. The compressor case is illustrated in Fig. 1 and the isentropic efficiency is defined by:

$$\eta_c = \frac{T_{out,s} - T_{in}}{T_{out} - T_{in}} \tag{3}$$

Likewise, the turbine case is illustrated in Fig. 2 and the isentropic efficiency is defined by:

$$n_t = \frac{T_{in} - T_{out}}{T_{in} - T_{out,s}} \tag{4}$$

The polytropic efficiencies for the compressor and the gas turbine are, in this paper, also defined on the basis of stagnation state conditions. The relationship between the polytropic efficiencies based upon static and stagnation states is discussed in detail in ref. [8]. The temperature–entropy diagrams are shown in Fig. 3 for the small compressor stage and in Fig. 4 for the small turbine stage. The polytropic efficiencies corresponding to Figs 3 and 4 respectively are:

$$\eta_{pc} = \frac{dT_i}{dT_a} \tag{5}$$

and

$$\eta_{pt} = \frac{dT_a}{dT_i} \tag{6}$$

The isentropic relationship for the elementary compressor is

$$\frac{dT}{T} = \frac{dT_i}{T} = (\gamma - 1)\frac{\gamma - 1\,dP}{\gamma P} \tag{7}$$

Combining equations (5) and (7) leads to:

$$\frac{dT_a}{T} = (\gamma - 1)\frac{\gamma - 1}{\gamma\eta_{pc}}\frac{dP}{P}$$

We may drop the subscript 'a' and integrate between the lower pressure and temperature and upper pressure and temperature of the compressor to give:

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{(\gamma-1)/\gamma\eta_{pc}} \tag{8}$$

In a somewhat similar way, equation (6) may be combined with the corresponding equation for the turbine:

$$\frac{dT_a}{T} = \frac{\gamma - 1}{\gamma}\eta_{pt}\frac{dP}{P} \tag{9}$$

and, after integration this leads to:

$$\frac{T_3}{T_4} = \left(\frac{P_3}{P_4}\right)^{\eta_{pt}(\gamma-1)/\gamma} \tag{10}$$

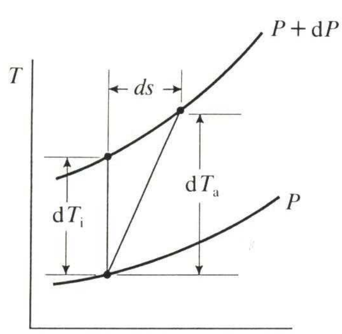

Fig. 3. Temperature–entropy diagram for an elementary compressor stage.

Before proceeding to the unified equations, it is appropriate to consider the limits of the efficiencies.

Limits of the efficiencies

The isentropic efficiencies introduced above are straightforward for the upper limit of the compressor and turbine and this corresponds to reversible isentropic cases which are

$$\eta_c = \eta_{pc} = 1.0 \text{ for the compressor}$$
$$\eta_t = \eta_{pt} = 1.0 \text{ for the turbine}$$

The lower limit is also straightforward in the case of the turbine, thus $\eta_t = \eta_{pt} = 0$ and this corresponds to an isothermal but adiabatic irreversible expansion.

The lower limit for the isentropic efficiency of the compressor is not straightforward and we need to formulate a clearer definition of a compressor.

The meaning of the verb to compress [9]—'To force into less space'—gives the key requirement, but we also have a primary requirement that the pressure should be increased, and the third condition, set down at the beginning of this section, was that the process should be adiabatic.

Some thought leads to the conclusion that the lower limiting case of the isentropic efficiency of an adiabatic compression process in which the pressure increases is one of constant specific volume. To find this, we seek the curve that joins the end state points of the fluid flowing through the compressor, which is the same as the curve of constant specific volume. That is:

$$\left(\frac{\partial T}{\partial s}\right)_{\eta_{pc}} = \left(\frac{\partial T}{\partial s}\right)_v \tag{11}$$

The constant volume expression is found using the property relationship $Tds = du + Pdv$ which, for a perfect gas becomes $Tds = C_v dT + Pdv$. This leads to:

$$\left(\frac{\partial T}{\partial s}\right)_v = \frac{T}{C_v} \tag{12}$$

The gradient of the compression curve on a temperature entropy diagram, with a polytropic compression η_{pc}, is given by:

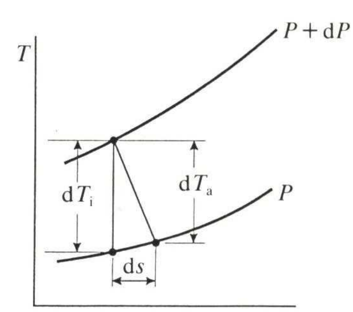

Fig. 4. Temperature–entropy diagram for an elementary turbine stage.

$$\left(\frac{\partial T}{\partial s}\right)_{\eta_{pc}} = \frac{dT_a}{ds}$$

Now the elementary change of specific entropy may be calculated from the gradient of a constant pressure curve together with the difference of the two differential temperatures, i.e.

$$ds = \left(\frac{\partial s}{\partial T}\right)_P (dT_a - dT_i)$$

The gradient of the constant pressure curve is found from the property relationship $Tds = dh - vdP$ which, for a perfect gas, becomes $Tds = C_p dT - vdP$ and this leads to:

$$\left(\frac{\partial s}{\partial T}\right)_P = \frac{C_p}{T}$$

Therefore, the elementary change of specific entropy required is:

$$ds = \frac{C_p}{T}(dT_a - dT_i)$$

and

$$\left(\frac{\partial T}{\partial s}\right)_{\eta_{pc}} = \frac{T}{C_p}\left(\frac{dT_a}{dT_a - dT_i}\right)$$

$$\therefore \left(\frac{\partial T}{\partial s}\right)_{\eta_{pc}} = \frac{T}{C_p}\left(\frac{1}{1-\eta_{pc}}\right) \quad (13)$$

Equations (12) and (13) may be inserted in equation (11) to give the minimum value of η_{pc} as $\eta_{pc,min}$, where

$$\frac{T}{C_v} = \frac{T}{C_p}\left(\frac{1}{1-\eta_{pc,min}}\right)$$

which reduces to $\eta_{pc,min} = (\gamma - 1)/\gamma$. This value may be used to calculate a minimum value of the isentropic efficiency of the compressor, thus:

$$\eta_{c,min} = \left[\frac{\left(\frac{P_2}{P_1}\right)^{(\gamma-1)/\gamma} - 1}{\left(\frac{P_2}{P_1}\right) - 1}\right] \quad (14)$$

Operation at a value of η_{pc} below $\eta_{pc,min}$

Operation of the machine at a value below $\eta_{pc,min}$ may at first be thought to be impossible. The machine could, however, be so inefficient that a state change could take place and the pressure could be increased to P_2 but the specific volume would increase so that $v_2 > v_1$ and the temperature would be:

$$T_2 > T_1\left(\frac{P_2}{P_1}\right)$$

but there would be no compression of the gas and the machine could not be said to be a 'compressor'. It could be called 'a non-compression pumping machine' and it may also be appropriate to call it a blower.

The term polytropic efficiency implies a polytropic process in which the parameter Pv^n does not change during the process. This type of relation can be applied to equations (8) and (10). The polytropic indices for the compressor n_c and the turbine n_t are given by:

$$n_c = \frac{\gamma\eta_{pc}}{1-\gamma(1-\eta_{pc})} \quad (15)$$

and

$$\eta_t = \frac{\gamma}{\gamma - \eta_{pt}(\gamma - 1)} \quad (16)$$

If, for a compressor, the concept of matching the inlet and outlet state properties with a parameter Pv^n was continued with beyond the constant specific volume mode of operation, the parameter n would suddenly switch from ∞ to $-\infty$ and the parameter could be written as n^1 where $n^1 = -n$ and the relationship would effectively change to P/v^{n^1}. The parameter n^1 would then be progressively reduced from ∞ to zero, at which $\eta_{pc} \to 0$. These relationships are illustrated on T–s and P–v diagrams in Fig. 5.

This has now set the scene to introduce the unified equations and the reciprocal rule for the equations.

UNIFIED EQUATIONS AND THE EFFICIENCY RECIPROCAL RULE

The usual equations for temperature ratio and pressure ratio in a compressor are:

$$\frac{T_2}{T_1} = \left\{1 + \frac{1}{\eta_c}\left[\left(\frac{P_2}{P_1}\right)^{(\gamma-1)/\gamma} - 1\right]\right\} \quad (17)$$

and

$$\frac{P_2}{P_1} = \left\{1 + \eta_c\left[\frac{T_2}{T_1} - 1\right]\right\}^{\gamma/(\gamma-1)} \quad (18)$$

The corresponding equations for the turbine are:

$$T_4 = T_3 - T_3\left[1 - \left(\frac{P_4}{P_3}\right)^{(\gamma-1)/\gamma}\right]\eta_t$$

which may be rearranged as:

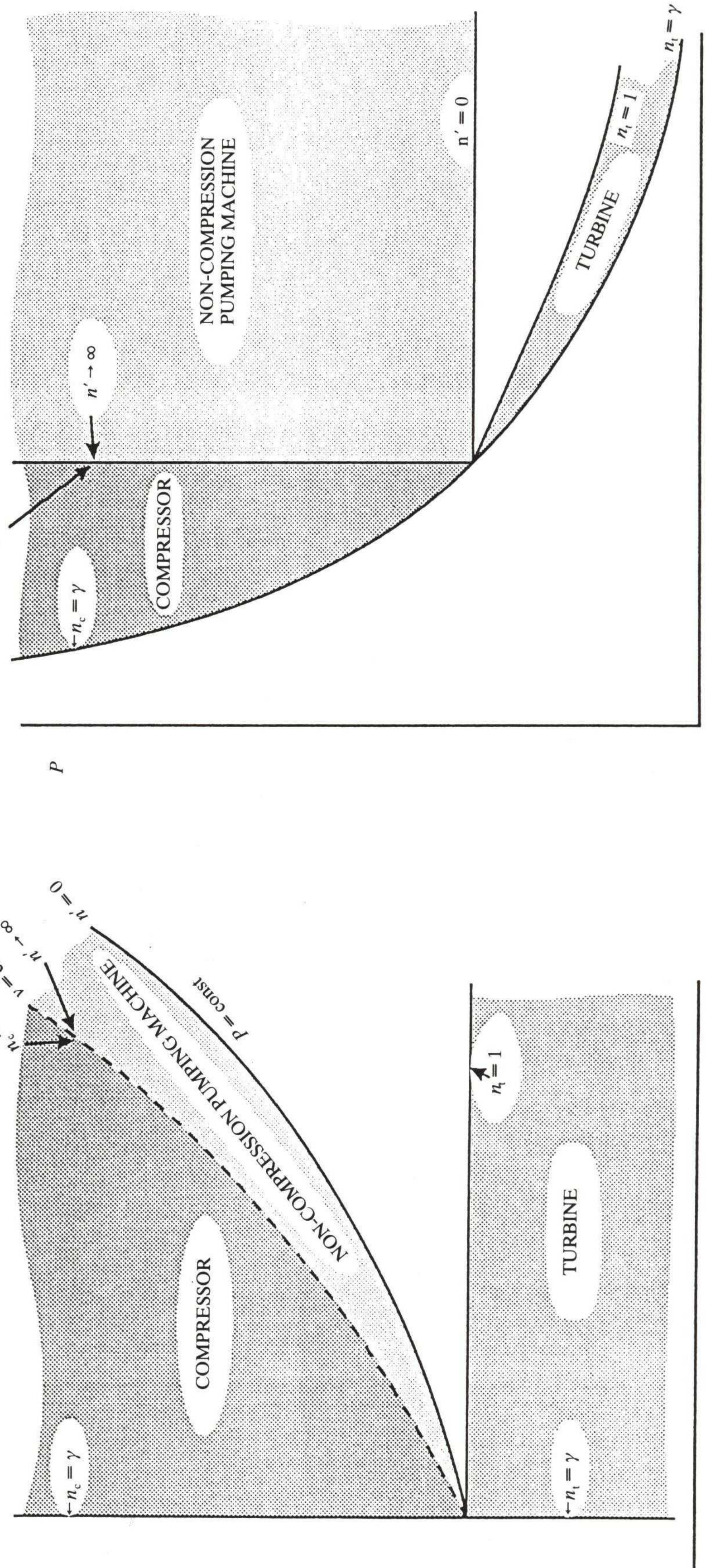

Fig. 5. Temperature–specific entropy and pressure–specific volume fields for compressors and turbines

$$T_4 - T_3 = T_3\eta_t\left[\left(\frac{P_4}{P_3}\right)^{(\gamma-1)/\gamma} - 1\right]$$

and

$$\frac{T_4}{T_3} = 1 + \eta_t\left[\left(\frac{P_4}{P_3}\right)^{(\gamma-1)/\gamma} - 1\right] \quad (19)$$

Likewise, it may be shown that

$$\frac{P_4}{P_3} = \left\{1 + \frac{1}{\eta_t}\left[\left(\frac{T_4}{T_3}\right) - 1\right]\right\}^{\gamma/(\gamma-1)} \quad (20)$$

If we now define a work ratio factor ξ by the equation:

$$\xi = \frac{T_{\text{out}} - T_{\text{in}}}{T_{\text{out,s}} - T_{\text{in}}} \quad (21)$$

here the 'out,s' state point is isentropically related to the inlet 'in' state point. Hence, we can write equations (17) and (19) as the single equation:

$$\frac{T_{\text{out}}}{T_{\text{in}}} = \left\{1 + \xi\left[\left(\frac{P_{\text{out}}}{P_{\text{in}}}\right)^{(\gamma-1)/\gamma} - 1\right]\right\} \quad (22)$$

Likewise, we may write equations (18) and (20) as the single equation:

$$\frac{P_{\text{out}}}{P_{\text{in}}} = \left\{1 + \frac{1}{\xi}\left[\frac{T_{\text{out}}}{T_{\text{in}}} - 1\right]\right\}^{\gamma/(\gamma-1)} \quad (23)$$

Thus, equations (21)–(23) are the first set of unified equations. To apply them to the compressor and the turbine, two operations are needed. First, the simple process of writing subscript $_1$ for 'in' and subscript $_2$ for 'out' for the compressor and subscript $_3$ for 'in' and subscript $_4$ for 'out' for the turbine. Secondly, the reciprocal rule has to be applied and this is simply that

$$\eta_c = \frac{1}{\xi} \quad (24)$$

and

$$\eta_t = \xi \quad (25)$$

The second set of unified equations also relate the pressure ratios and temperature ratios across the turbo-machines. The term polytropic efficiency implies that the end state points of the turbo-machine can be fitted with a parameter, of the form Pv^n, which does not change between the inlet and outlet state points. This leads to:

$$\frac{T_{\text{out}}}{T_{\text{in}}} = \left(\frac{P_{\text{out}}}{P_{\text{in}}}\right)^{(n-1)/n} \quad (26)$$

and this is also related to:

$$\frac{T_{\text{out}}}{T_{\text{in}}} = \left(\frac{P_{\text{out}}}{P_{\text{in}}}\right)^{\xi_p(\gamma-1)/\gamma} \quad (27)$$

here ξ_p is defined as a polytropic or small stage work ratio, defined as:

$$\xi_p = \left(\frac{dT_a}{dT_i}\right) \quad (28)$$

in a manner similar to that introduced earlier.

It may be shown that these equations may be applied to the compressor and turbine by using the above subscript transformations and a second similar reciprocal rule, as follows:

$$\eta_{pc} = \frac{1}{\xi_p} \quad (29)$$

and

$$\eta_{pt} = \xi_p \quad (30)$$

It also follows that:

$$n = \frac{\gamma}{\gamma - \xi_p(\gamma - 1)} \quad (31)$$

It is interesting to note that for a fixed ξ_p or n, the compression or expansion process, represented by equations (26) or (27), is a straight line in the logarithmic temperature–entropy diagram [10]; this (LTS) is briefly reviewed in Appendix 1.

The results of the above discussion are summed up using equations (29)–(31), in Fig. 6. This is clearly illustrated, in terms of n, for the turbine and the compressor region. It also shows very clearly the non-zero lower limit of the polytropic efficiency of a compressor and the asymptotes for η_{pc} and ξ_p as n approaches infinity. The range and the limiting values of various parameters are shown in Table 1.

DISCUSSION

There are many similarities and differences between compressors and turbines and the relationship has positive and negative features, reciprocal aspects and other features that directly correspond to one another. Examples of these, respectively, are the work or power interactions, the ratio of ideal to actual work interactions and the increase of specific entropy of the gas flowing through the machine.

However, certain aspects of these may be likened to a complex conjugate pair on an Argand diagram. The temperature–specific entropy diagram shown in Fig. 1 for the compressor has some resemblance to a mirror image of the corresponding diagram shown in Fig. 2 for the turbine. The temperature and pressures for the compressor and turbine

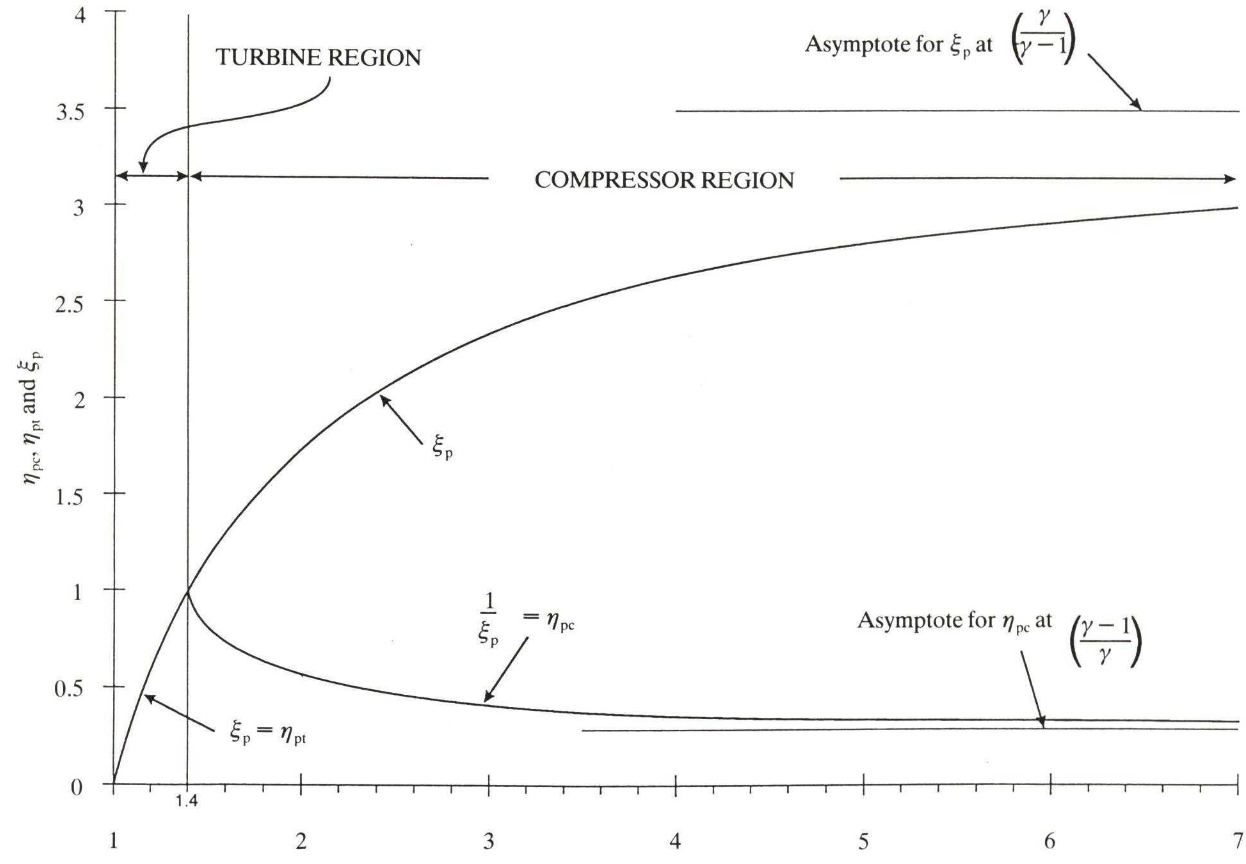

Fig. 6. Polytropic efficiencies of compressor and turbine as functions of polytropic index.

Table 1. Range and limiting values of polytropic efficiencies and work ratio

n and n^1	$\eta_{pt} = \xi_p$	$\eta_{pc} = \frac{1}{\xi_p}$	$\eta_{ppm} = \frac{1}{\xi_p}$	
$n = 1$	0			lower limit for turbine
$1 < n < \gamma$	$0 < \eta_{pt} < 1.0$			normal range for turbine
$n = \gamma = 1.4$	1.0	1.0		upper limit for turbine and compressor
$\gamma < n < \infty$		$\frac{\gamma-1}{\gamma} < \eta_{pc} < 1.0$		normal range for compressor
$n \rightarrow \infty$		$\eta_{pc} \rightarrow \frac{\gamma-1}{\gamma}$		lower limit for compressor
$n \rightarrow -\infty$ $n^1 \rightarrow \infty$				limiting condition for a non-compression pumping machine, changeover from n to n^1
$0 < n^1 < \infty$			$0 < \eta_{ppm} < \frac{\gamma-1}{\gamma}$	range for non-compression pumping machine

Equations:
For compressor and turbines $\frac{P}{Ⓟ}\left(\frac{v}{ⓥ}\right)^n = 1$

For a non-compression pumping machine $\frac{P}{Ⓟ} = \left(\frac{v}{ⓥ}\right)^{n^1}$

move in opposite directions and yet the specific entropy for both machines move in the direction of an increase. These aspects are discussed in some detail in Appendix 2.

The definitions of the isentropic efficiency for the two machines have a reciprocal relationship only because we have elected to define them in the way we do and then call them efficiencies. Once we have called a parameter of assessment an efficiency, it follows that we expect the upper limit not to exceed unity or 100% and, in some cases, e.g. a Carnot cycle, the upper limit is considerably less than unity. However, we do not normally expect an efficiency of zero to be beyond reach.

If, as in equation (28), we define ξ as a *work ratio*, which is the actual divided by ideal, and not call it an efficiency, as in the turbine case, the reciprocal rule is not needed. We can still envisage a work ratio with a value greater than unity for a compressor and less than one for a turbine, without any difficulties.

The unified equations (22) and (23) relate the temperature ratio and the pressure ratio, respectively, for both the compressor and turbine.

The corresponding unified equations (26) and (27) also relate pressure and temperature ratios. Equation (26) is based upon the polytropic exponent n and a diagram illustrating the role of it in T–s and P–v fields is shown in Fig. 5. The region applicable to the non-compression pumping machine is also included on this diagram. Equation (28) is based upon the small stage work ratio, which corresponds to the full work ratio discussed above. The reciprocal rule introduced for transforming the isentropic efficiency from the compressor to the turbine is also applicable for the polytropic efficiencies, as shown in equations (29) and (30). The polytropic index is shown as a function of the polytropic work ratio in equation (31).

The material presented here is intended to provide increased understanding of the relationships between compressors and turbines. This, together with the unified equations derived, is of use not only in understanding but also in writing computer software for turbo-machinery, where the unified equations can lead to common algorithms for both compressors and turbines.

CONCLUSIONS

1. The relevance of the work to engineering education has been explained in some detail in a foreword to the paper.
2. The similarities and the differences between compressors and turbines has been identified, discussed and sets of unified equations have been developed.
3. The study has provided further fundamental understanding of the processes and relationships for turbines and compressors.
4. The unified equations developed will be used to computer modellers in writing software for simulating turbo-machinery. Common algorithms based upon the unified equations can be used for both compressors and turbines.
5. It has been discovered that a turbo-compressor has a non-zero minimum polytropic efficiency, which is given by:

$$\eta_{\mathrm{pc,min}} = \frac{\gamma - 1}{\gamma}$$

and a corresponding minimum isentropic efficiency given by:

$$\eta_{\mathrm{c,min}} = \left[\frac{(P_2/P_1)^{(\gamma-1)/\gamma} - 1}{(P_2/P_1) - 1}\right]$$

6. If a machine is operated at a polytropic or isentropic efficiency below the value given in conclusion (5), no compression of the gas takes place and the machine is therefore not operating as a compressor. It may be operating as a blower or a non-compression pumping machine.
7. The range of values of the polytropic exponent has been determined for operation of a turbo-machine as a compressor and as a turbine.
8. A logarithmic temperature—entropy diagram, disccussed by the late Professor W. J. Kearton, has been reviewed and shown to be a most useful diagram.
9. It is claimed that compressors and turbines are conjugate machines. The conditions required for the outlet state points for a compressor and a turbine to be a true complex conjugate pair have been disussed in detail in Appendix 2.

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APPENDIX 1: THE LOGARITHMIC TEMPERATURE–ENTROPY DIAGRAM

The author was first introduced to the logarithmic temperature–entropy diagram as an undergraduate by the late Professor W. J. Kearton [10]. As it does not seem to be in current use, a brief review is given here.

We may apply a corollary of the first law of thermodynamics to an elementary step, to obtain:

$$\delta_q + \delta_w = \delta_u$$

Here the new notation recommended by Mayhew [6] has been used. The symbols are:

δq elementary heat transfer to system
δ_w elementary work transfer to the system
δ_u elementary increase in internal energy of the system

Next, the heat and work transfer processes are considered to be reversible and, accordingly, we may write:

$$\delta_q = T\delta s$$
$$\delta_w = -P\mathrm{d}v$$

considering the system as a perfect gas, we may also write:

$$\delta u = C_v \delta T$$

and, substituting, we obtain

$$T\delta s = C_v \delta T + P\mathrm{d}v$$

We note that all the terms of this equation are properties of the system of unit mass and that we may proceed to the limit and write them as differentials, thus:

$$T\mathrm{d}s = C_v \mathrm{d}T + P\mathrm{d}v \tag{A1}$$

Using the relations $Pv = RT$, $C_p - C_v = R$ and $C_p/C_v = \gamma$, we may readily derive

$$\frac{\mathrm{d}s}{C_p} = \frac{1}{\gamma}\frac{\mathrm{d}T}{T} + \frac{(\gamma-1)}{\gamma}\frac{\mathrm{d}v}{v} \tag{A2}$$

and

$$\frac{\mathrm{d}s}{C_p} = \frac{\mathrm{d}T}{T} - \frac{(\gamma-1)}{\gamma}\frac{\mathrm{d}P}{P} \tag{A3}$$

Equations (A2) and (A3) may be integrated from a reference state, which can be defined as properties $\textcircled{s}$ - $\textcircled{T}$ - $\textcircled{P}$, and $\textcircled{v} = 1/\textcircled{\rho}$.

The results may be expressed in terms of non-dimensional parameters as:

$$\ln\frac{T}{\textcircled{T}} = \gamma\left(\frac{s-\textcircled{s}}{C_p}\right) + (\gamma-1)\ln \textcircled{v}/v \tag{A4}$$

and we note that $\textcircled{v}/v = \rho/\textcircled{\rho}$ and

$$\ln\frac{T}{\textcircled{T}} = \frac{s-\textcircled{s}}{C_p} + \frac{\gamma-1}{\gamma}\ln\frac{P}{\textcircled{P}} \tag{A5}$$

We can compare these two equations with the straight line $Y = mX + c$ where, for equation (A4)

$$Y = \ln T/\textcircled{T}$$
$$m = \gamma$$
$$X = \frac{s-\textcircled{s}}{C_p}$$

$$c = (\gamma - 1)\ln v/v = (\gamma - 1)\ln \rho/\textcircled{\rho}$$

This represents a family of lines of constant specific volume or constant density ratio. We have, for equation (A5),

$$Y = \ln T/\textcircled{T}$$
$$m = 1$$
$$X = \frac{s - \textcircled{s}}{C_p}$$
$$c = \frac{(\gamma - 1)}{\gamma}\ln P/\textcircled{P}$$

This represents a family of lines of constant pressure ratio.

Equations (A4) and (A5) are illustrated graphically in Fig. 7. It is convenient to call this the logarithmic temperature–entropy diagram, i.e. the LTS diagram.

A polytropic process represented by the equation

$$\frac{P}{\textcircled{P}} = \left(\frac{\textcircled{v}}{v}\right)^n \qquad \text{(A6)}$$

or

$$\frac{P}{\textcircled{P}} = \left(\frac{\rho}{\textcircled{\rho}}\right)^n$$

may readily be shown to be represented by the equation:

$$\ln\frac{T}{\textcircled{T}} = \frac{\gamma(n-1)}{(n-\gamma)}\left(\frac{s - \textcircled{s}}{C_p}\right) \qquad \text{(A7)}$$

This represents a series of straight lines through the origin and is shown in the inset on Fig. 7. The implication of this is that the locus of the state points represented by equation (26) for a fixed n, or by equation (27) for a fixed value of the polytropic work ratio ξ_p, is a straight line in the LTS diagram.

APPENDIX 2: A COMPRESSOR AND TURBINE AS A CONJUGATE PAIR

A complex conjugate pair, Z_1 and Z_2 illustrated in Fig. 8, may be represented by

$$Z_1 = \mathrm{Re}^{i\Phi} = a(\cos\Phi + i\sin\Phi) \qquad \text{(A8)}$$

$$Z_2 = \mathrm{Re}^{-1\Phi} = a(\cos\Phi - i\sin\Phi) \qquad \text{(A9)}$$

In the following part of this appendix, it is shown that for a turbine and a compressor working on a perfect gas with a fixed γ, which is the same for both machines, it is possible to find a range of pressure ratios for the compressor and the turbine and maximum to minimum temperature ratios, which are related to one another, so that the compressor and turbine represent a true conjugate pair of machines. This means that the temperature rise in the compressor is numerically equal to the temperature drop in the turbine, and also that the increase in specific entropy in the compressor is equal to the specific entropy increase in the turbine.

The entropy change is considered first. Using the equations given in Appendix 1, the entropy change in the the compressor may be expressed as:

$$\frac{s_2 - s_1}{C_p} = \ln\frac{T_2}{T_1} - \left(\frac{\gamma - 1}{\gamma}\right)\ln\frac{P_2}{P_1} \qquad \text{(A10)}$$

and, likewise, the entropy change in the turbine may be expressed as:

$$\frac{s_4 - s_3}{C_p} = \ln\frac{T_4}{T_3} - \left(\frac{\gamma - 1}{\gamma}\right)\ln\frac{P_4}{P_3} \qquad \text{(A11)}$$

The specific entropy change in the compressor and the specific entropy change in the turbine are equal, hence:

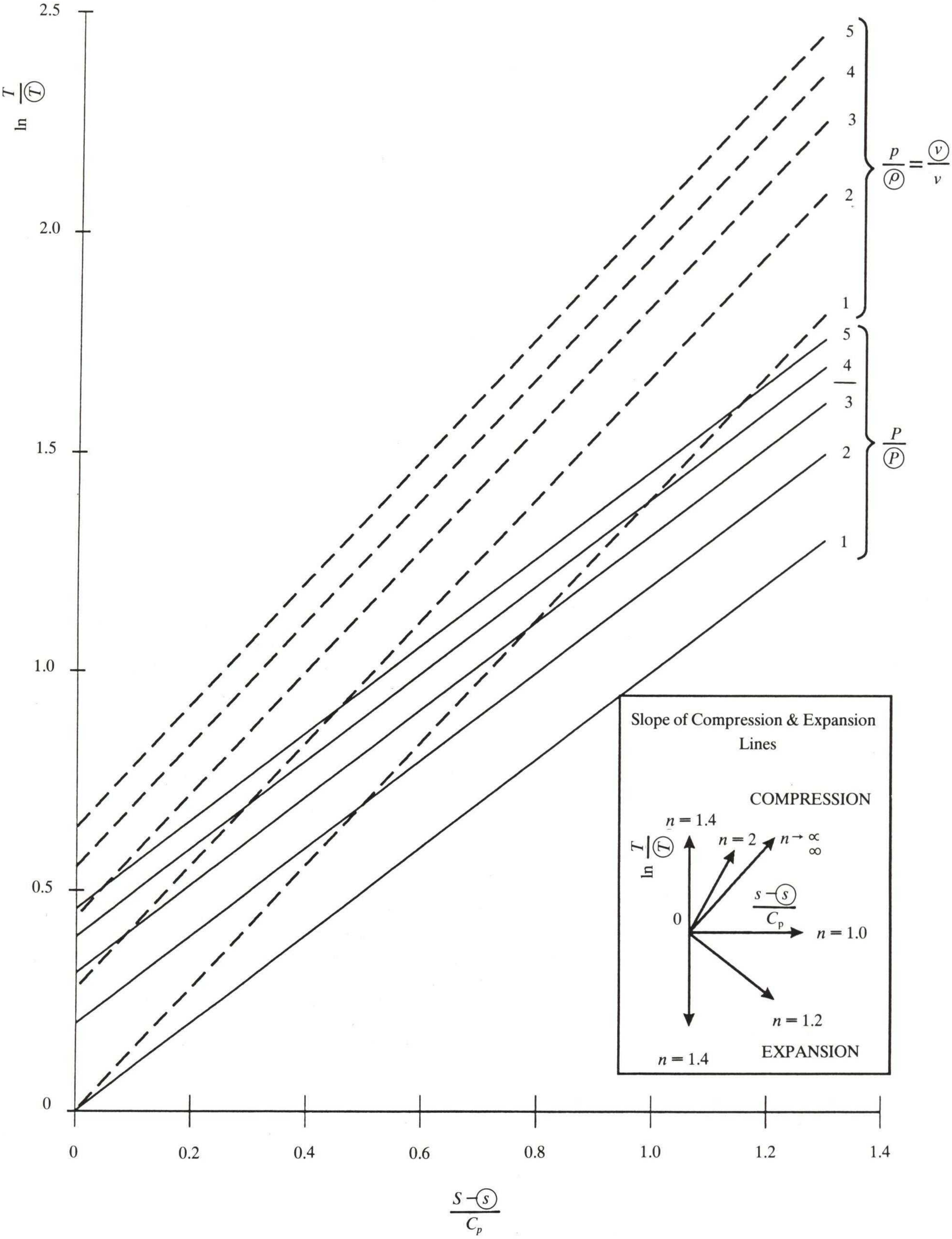
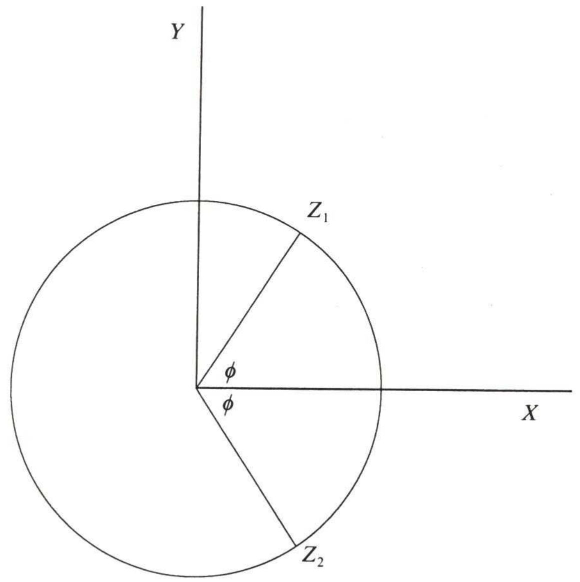


Fig. 7. Logarithm temperature ratio–entropy diagram showing lines of constant pressure ratios and volume ratios.



$Z_1 = a(\mathrm{Cos}\phi + i\,\mathrm{Sin}\phi)$
$Z_2 = a(\mathrm{Cos}\phi - i\,\mathrm{Sin}\phi)$

Fig. 8. Illustration of a complex conjugate pair.

$$\ln\left[\frac{T_2}{T_1}\left(\frac{P_2}{P_1}\right)^{-[(\gamma-1)/\gamma]}\right] = \ln\left[\frac{T_4}{T_3}\left(\frac{P_4}{P_3}\right)^{-[(\gamma-1)/\gamma]}\right]$$
$$\therefore \frac{T_2}{T_1}\left(\frac{P_2}{P_1}\right)^{-[(\gamma-1)/\gamma]} = \frac{T_4}{T_3}\left(\frac{P_4}{P_3}\right)^{-[(\gamma-1)/\gamma]} \qquad \text{(A12)}$$

If we write

$$x = \left(\frac{P_2}{P_1}\right)^{(\gamma-1)/\gamma}$$

and

$$y = \left(\frac{P_4}{P_3}\right)^{-[(\gamma-1)/\gamma]}$$

then

$$\frac{T_2}{T_1}\frac{1}{x} = \frac{T_4}{T_3}y \text{ or } xy = \frac{T_2}{T_1}\frac{T_3}{T_4} \qquad \text{(A13)}$$

In the following part of this appendix, it is shown that it is possible, in principle, to find a compressor in which the temperature rise is equal in magnitude to the temperature drop in a turbine, and also that the increase in specific entropy in the same compressor is equal to the specific entropy increase in the same turbine.

The temperature changes are now considered.

For the compressor we have:

$$\frac{T_2 - T_1}{T_1} = \frac{x - 1}{\eta_c} \qquad \text{(A14)}$$

and for the turbine we have

$$\frac{T_3 - T_4}{T_3} = \frac{y-1}{y}\eta_t \tag{A15}$$

If the temperature drop in the turbine is normalized to the same reference as that of the compressor, thus:

$$\frac{T_3 - T_4}{T_1} = \frac{T_3}{T_1}\left(\frac{\gamma-1}{\gamma}\right)\eta_t \tag{A16}$$

We require the temperature rise in the compressor to equal the temperature drop in the turbine. Therefore:

$$\frac{T_2 - T_1}{T_1} = \frac{T_3 - T_4}{T_1}$$

Hence

$$\frac{x-1}{\eta_c} = \frac{T_3}{T_1}\left(\frac{y-1}{y}\right)\eta_t \tag{A17}$$

Next, we may find expressions for the temperature ratios T_2/T_1 and T_3/T_4 in terms of the component efficiencies and isentropic temperature ratios, using equations (A14) and (A15). This is carried out as follows:

$$\frac{T_2}{T_1} = \left[1 + \left(\frac{x-1}{\eta_c}\right)\right] \tag{A18}$$

and

$$\frac{T_4}{T_3} = \left[1 - \left(\frac{y-1}{y}\right)\eta_t\right] \tag{A19}$$

Combining equations (A13), (A18), and (A19), we obtain:

$$y\left[1 - \left(\frac{y-1}{y}\right)\eta_t\right] = \frac{1}{x}\left[1 + \frac{x-1}{\eta_c}\right] \tag{A20}$$

$$\therefore y = \frac{\frac{1}{x}\left[1 + \frac{(x-1)}{\eta_c}\right] - \eta_t}{(1-\eta_t)} \tag{A21}$$

Hence the requirement that the specific entropy change in the compressor is equal to the specific entropy change in the turbine leads to a relationship between x, y, η_c and η_t shown by equation (A21). The relationship between x and y is shown graphically for $\eta_c = 0.8$ and $\eta_t = 0.9$ in Fig. 9 and the corresponding relationship between pressure ratios P_2/P_1 and P_3/P_4 are shown in Fig. 10.

It is interesting to note that for

$$1 < P_2/P_1 < 24.7$$

then

$$P_3/P_4 > P_2/P_1$$

but for

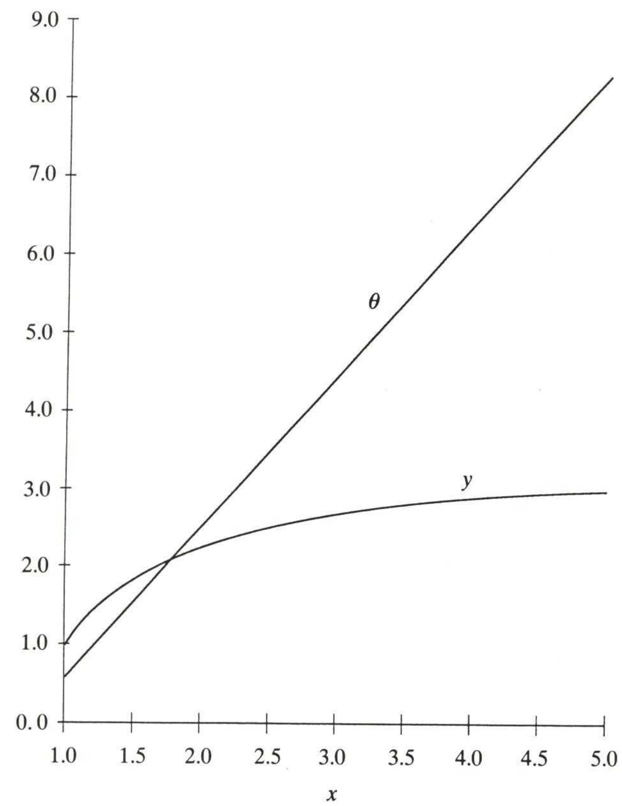
$$P_2/P_1 > 24.7$$

then

$$P_2/P_1 > P_3/P_4$$

The changeover point occurs at $x = y = 2.5$.

The requirement that the temperature rise in the compressor is numerically equal to the temperature drop in the turbine gave rise to equation (A17); this may be expressed as:



$\theta = \frac{T_3}{T_1}$ Temperature ratio

$x = \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}}$ Isentropic temperature ratio for compressor

$y = \left(\frac{P_3}{P_4}\right)^{\frac{\gamma-1}{\gamma}}$ Isentropic temperature ratio for turbine

Fig. 9. Relationship of θ and y as functions of x for compressor and turbine to be a true conjugate pair.

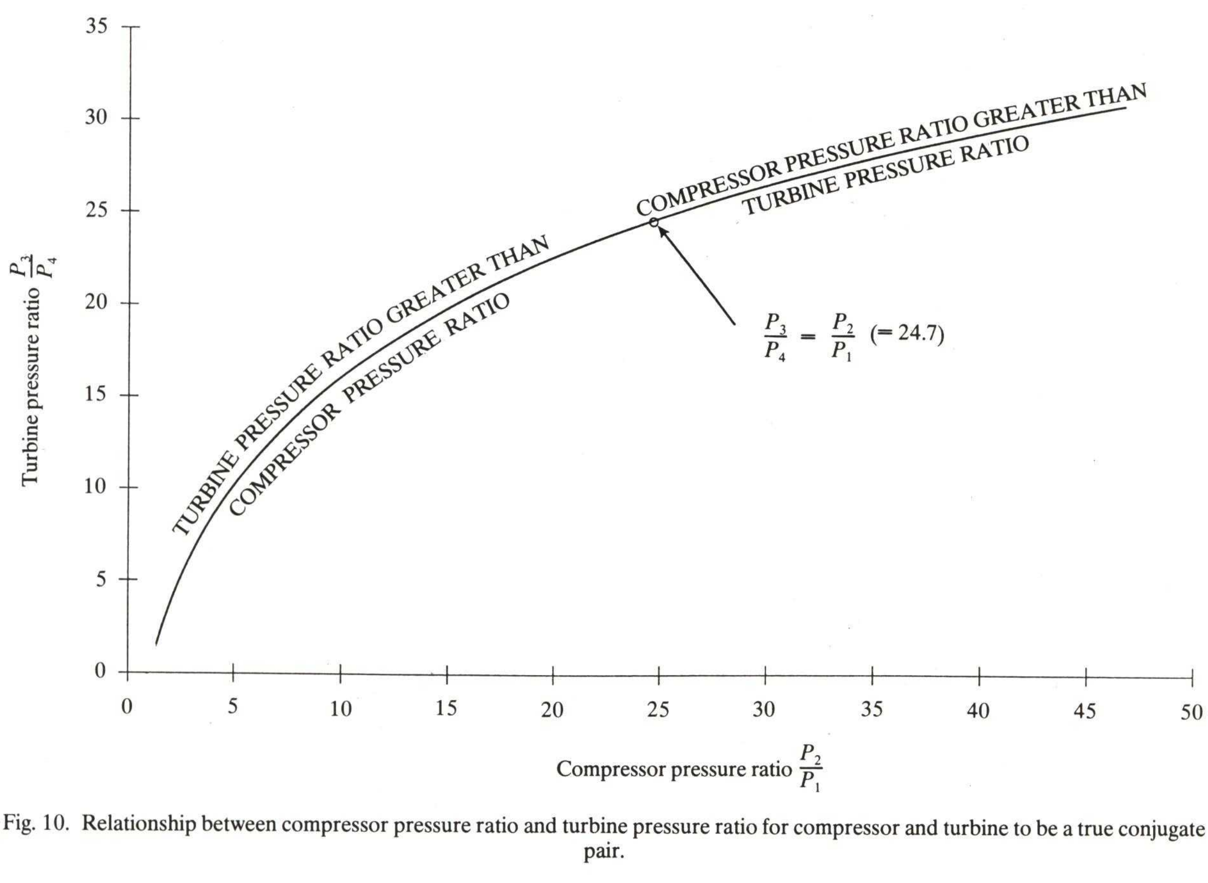


Fig. 10. Relationship between compressor pressure ratio and turbine pressure ratio for compressor and turbine to be a true conjugate pair.

$$\frac{T_3}{T_1} = \frac{(x-1)}{\eta_c \eta_t (1 - 1/y)} \tag{A22}$$

or

$$\theta = \frac{(x-1)}{\eta_c \eta_t (1 - 1/y)} \tag{A23}$$

This relationship is shown graphically in Fig. 9. The changeover point, mentioned above, occurs when

$$\theta = \frac{2.5}{\eta_c \eta_t} = 3.4722 \tag{A24}$$

The conjugate points are first given by

$$\frac{T_2 - T_1}{T_1} = \frac{x-1}{\eta_c} = a \sin\Phi \tag{A25}$$

and

$$\frac{s_2 - s_1}{C_p} = \ln\left[\left(1 + \frac{x-1}{\eta_c}\right)\frac{1}{x}\right] = a \cos\Phi \tag{A26}$$

for the compressor.

Secondly,

$$\frac{T_4 - T_3}{T_1} = -\frac{T_3}{T_1}\left(\frac{y-1}{y}\right)\eta_t = -a \sin\Phi \tag{A27}$$

and

$$\frac{s_4 - s_3}{C_p} = \ln\left(\frac{T_4}{T_3}\right) + \ln\left(\frac{P_3}{P_4}\right)^{(\gamma-1)\gamma} = a\cos\Phi$$

$$\ln\left[\left(1 - \frac{y-1}{y}\eta_t\right)y\right] = a\cos\Phi \tag{A28}$$

for the turbine. It therefore follows that:

$$\tan\Phi = \frac{(x-1)}{\eta_c \ln\left[\left(1 + \frac{x-1}{\eta_c}\right)\frac{1}{x}\right]} \tag{A29}$$

At the changeover point, the values are: $a = 1.88$ and $\Phi = 85.73°$.

The conclusions from this appendix are that it is possible to find a compressor and turbine that truly represent a complex conjugate pair. In the non-dimensional plane of temperature change and specific entropy change, for reasonable values of isentropic efficiency of the compressor and turbine there are a range of values of turbine pressure ratios (P_3/P_4) and temperature ratios θ which correspond to a range of compressor pressure ratios P_2/P_1.

Finally, the corresponding pressure ratios for the turbine, for a value of $\gamma = 1.4$, are greater than those of the compressor in the range of $P_2/P_1 = 1$ to $P_2/P_1 = 24.7$ and less than those of the compressor in the range of $P_2/P_1 > 24.7$.

Professor Woods, a graduate of the University of Liverpool, is a Fellow of the Royal Academy of Engineering, and a Fellow of the Institution of Mechanical Engineers. He has worked at Rolls Royce Ltd, and the Universities of Liverpool and London. He has been a Visiting Professor at the Massachusetts Institute of Technology and a Visiting Professor at the

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