Int. J. Engng Ed. Vol. 8, No. 1, pp. 71–75, 1992
Printed in Great Britain.

0742–0269/91 $3.00+0.00

Exact Evaluation of Internal Forces for Beam Elements Carrying Uniformly Distributed Loads*

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In a classical 2D truss/beam finite element, based on the technical theory of beams, the axial and transverse displacement fields are normally interpolated with linear and cubic shape functions respectively. These elements provide exact results when modelling rigid joined structures with concentrated forces and moments. It is shown, within the framework of finite elements formulation and with the aid of additional shape functions, that the nodal displacement solution is exact when uniformly distributed loads are applied. Using these additional shape functions a method is developed to obtain the exact shear forces and bending moments in such loading conditions.

INTRODUCTION

SEVERAL introductory finite element teaching programs in structural analysis [1–4] present the truss and beam elements and apply them to the analysis of planar frames, including the evaluation of shear forces and bending moments at nodal points.

It is well known that models of planar structures using the standard beam element give exact displacements and generalized forces at nodal points provided only concentrated loads are applied. When uniformly distributed loads are considered, nodal equivalent point forces can be calculated from the potential terms of such loads and exact nodal displacements are still obtained. However, if shear forces and bending moments are evaluated by derivation of the displacement field, strong inaccuracies are expected. This is due to the fact that the standard beam element cannot model exactly the fourth order transverse displacement fields and the second order axial displacement fields which occur in such loading conditions.

Based on different physical arguments, the evaluation of internal forces for elements carrying uniformly distributed loads has been presented in the literature in several ways.

Nath [5] suggests that the nodal equivalent point forces, being merely an expedient for calculating exact nodal displacements, cannot be strictly considered as external loads. Consequently these forces must be subtracted to the internal generalized forces obtained from the derivation of the displacement fields. Coates *et al.* [6] suggest a two step analysis: in the first step a full stiffness analysis is carried out to render the structure 'kinematically determinate' by clamping all joints against displacement and a 'particular' solution is obtained. In the second step a 'complementary' analysis is carried out using a set of loads equal and opposite to the fixed end moments and forces exerted by the clamps in step 1. The solution to the problem is then obtained by superimposing the 'particular' and the 'complementary' solutions.

In both methods internal forces $\mathbf{F}^e$ are evaluated for each member using the following expression

$$\mathbf{F}^e = \mathbf{K}^e\mathbf{q}^e - \mathbf{Q}^e \tag{1}$$

where $\mathbf{K}^e$ is the element stiffness matrix, $\mathbf{q}^e$ the element nodal displacement solution and $\mathbf{Q}^e$ a correction term corresponding to the nodal equivalent point loads associated with the distributed loads.

These methods have undoubtedly the advantage of providing some insight into physical aspects of this problem. However questions regarding the exact displacement solution obtained in such loading situations and the role played by the correction term $\mathbf{Q}$ are yet to be answered within the framework of the finite element formulation.

For the sake of completeness a brief outline of the finite element formulation is presented.

The truss and beam elements are developed introducing additional higher order shape functions to model the displacement fields associated with uniformly distributed loads. It will be seen why in such loading conditions the displacement solution is exact for these elements. The formal reason for the introduction of a correction term $\mathbf{Q}^e$ in the evaluation of internal forces as shown in expression (1) is also presented.

* Paper accepted 20 November 1991.
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THE FINITE ELEMENT METHOD

The application of the finite element method in linear elastic structural analysis can be briefly described in the following major steps:

Step 1

A structure/continuum is divided into elements; in each element the displacement field is chosen in terms of a set of shape functions and a set of nodal displacement parameters.

Step 2

Based on the principle of minimum total potential energy, a stiffness matrix relating the nodal forces and displacements is obtained for each element from the strain energy terms. Also a load vector is obtained from the potential of the external forces. For each element an equilibrium equation can be written in the form

$$\mathbf{K}^e\mathbf{q}^e = \mathbf{f}^e \tag{2}$$

where $\mathbf{K}^e$ is the stiffness matrix, $\mathbf{q}^e$ is the vector of the nodal generalized displacements and $\mathbf{f}^e$ is the generalized force vector.

Step 3

A global equilibrium equation

$$\mathbf{K}^g\mathbf{q}^g = \mathbf{f}^g \tag{3}$$

is obtained by assembling a global stiffness matrix $\mathbf{K}^g$ and in a global load vector $\mathbf{f}^g$ the different element stiffness matrices and the corresponding element load vectors, respectively, according to the topological description of the structure's finite element model. $\mathbf{q}^g$ is the vector of global degrees of freedom.

Step 4

Once the displacement solution of equation (3) is obtained, internal forces can be calculated in each element by proper derivation of the displacement fields.

The selection of displacement functions is a crucial step on the finite element formulation since it determines the ability of an element to accurately model the expected displacement fields within the structure.

THE TRUSS/BEAM ELEMENT

The standard truss/beam element has two extreme nodes, six degrees of freedom, two translations and one rotation in each node as illustrated in Fig. 1.

This element can be obtained by the superposition of the beam element and the truss element. These two simpler elements are now described.

Truss element

Consider the one dimensional truss element. This element has two axial degrees of freedom: u_1 and u_2 as shown in Fig. 2.

If the element is carrying a uniformly distributed load Q_x it is well known from elasticity theory that the displacement u is quadratic in x.

Consider the displacement within the element to be given by

$$u = N_1u_1 + N_2u_2 + N_3p \tag{4}$$

where p is an extra degree of freedom and

$$N_1 = (L-x)/L; \qquad N_2 = x/L; \qquad N_3 = N_1N_2 \tag{5}$$

This element now has an extra quadratic shape function as compared to the standard truss element. It can be seen that $N_3 = 0$ for $x = 0$ and $x = L$, thus u_1 and u_2 can still be interpreted as the nodal axial displacements.

The strain energy of the element is given by

$$U = \frac{1}{2}\int_0^L EA\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x \tag{6}$$

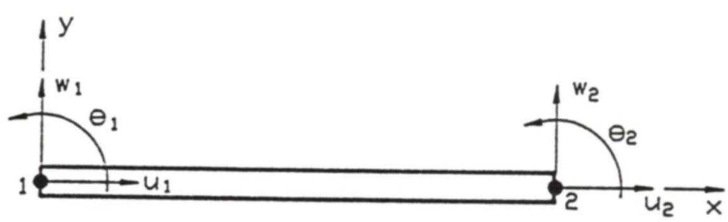

Fig. 1. Truss/beam element.

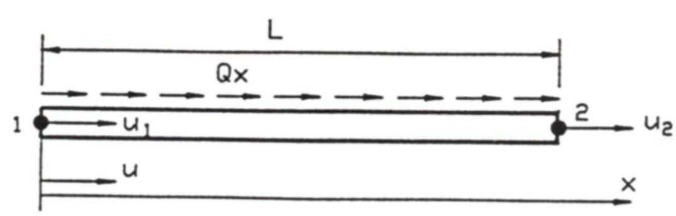

Fig. 2. Truss element.

where E and A are the Young's modulus and the truss cross-sectional area, respectively.

If expression (4) is derived with respect to x and substituted in (6) the element stiffness matrix is obtained

$$\mathbf{K}_i^e = \int_0^L \frac{EA}{L}\,[N_1' N_2' N_3']^T [N_1' N_2' N_3']\,\mathrm{d}x$$

$$= \frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \quad (7)$$

the potential of the distributed load Q_x is defined by

$$\Omega_q = \int_0^L Q_x u\,\mathrm{d}x \quad (8)$$

and the nodal equivalent point forces f_i^e can be calculated

$$f_{t_i}^e = \int_0^L Q_x N_i\,\mathrm{d}x \quad (9)$$

or

$$\mathbf{f}_i^e = [Q_x L/2; \quad Q_x L/2; \quad Q_x L/6]^T \quad (10)$$

The equilibrium equation for the element can be established

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ p \end{Bmatrix} = Q_x L \begin{Bmatrix} 1/2 \\ 1/2 \\ 1/6 \end{Bmatrix} \quad (11)$$

The extra degree of freedom p is not coupled with u_1 and u_2 and can be immediately calculated: $p = Q_x L^2/2EA$; this result shows that u_1 and u_2 are exact for the present loading conditions even assuming a linear displacement field.

If, for example, $u_2 = 0$ (which correspond to the 1D problem of a simply supported rod subjected to its own weight), equation (11) has the solution

$$u_1 = \frac{Q_x L^2}{2EA}; \; u_2 = 0; \; p = \frac{Q_x L^2}{2EA} \quad (12)$$

The internal forces in the element can be evaluated

$$F_1 = -AE\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=0}; \; F_2 = AE\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=L} \quad (13)$$

or in matrix form

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = AE\begin{bmatrix} -N_1'|_{x=0} & -N_2'|_{x=0} \\ -N_1'|_{x=L} & -N_2'|_{x=L} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$+ EAp\begin{Bmatrix} -N_3'|_{x=0} \\ -N_3'|_{x=L} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$+ \frac{Q_x L}{2}\begin{Bmatrix} -1 \\ -1 \end{Bmatrix} \quad (14)$$

substituting the solution (12) in equation (14)

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} Q_x L/2 \\ -Q_x L/2 \end{Bmatrix} + \begin{Bmatrix} -Q_x L/2 \\ -Q_x L/2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -QL \end{Bmatrix} \quad (15)$$

which is the exact solution.

Equation (15) suggests another form

$$\mathbf{F}_i^e = \mathbf{K}^e \mathbf{q}_i^e - \mathbf{Q}^e \quad (16)$$

where $\mathbf{q}_i^e$ is the element displacement solution.

The vector $\mathbf{Q}_i^e$ is a correction term that corresponds to the nodal point loads equivalent to the distributed forces Q_x.

In Fig. 3 the exact displacement solution is compared with the finite element results.

It can be observed that the corrective term $Q_x L/2$ in expression (15) corresponds, in terms of derivatives of u, to measure the slope of angles $\alpha + \beta_1$, and $\alpha + \beta_2$ at node points 1 and 2 respectively, instead of α alone as illustrated in Fig. 3.

Comparing expressions (7) and (14) it can be seen that

$$\int_0^L N_1' N_1'\,\mathrm{d}x = -N_1'|_{x=0} \quad (17)$$

$$\int_0^L N_1' N_2'\,\mathrm{d}x = -N_2'|_{x=0} = N_1'|_{x=L} \quad (18)$$

$$\int_0^L N_2' N_2'\,\mathrm{d}x = -N_2'|_{x=0} = N_1'|_{x=1} \quad (19)$$

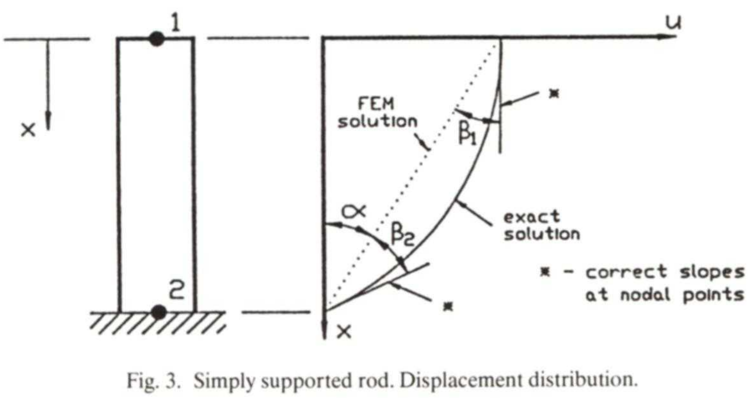

Fig. 3. Simply supported rod. Displacement distribution.

Considering, for instance, expression (18) and integrating the left hand side by parts

$$\int_0^L N_1' N_2' \mathrm{d}x = N_1 N_2|_0^L - \int_0^L N_1' N_2'' \mathrm{d}x \quad (20)$$

N_2 is linear in x thus $N_2'' = N_1|_{x=L} = 0$ thus proving the relationship (18).

Expressions (17) and (19) can be equally demonstrated in the same manner.

Beam element

Consider the 4DOF beam element shown in Fig. 4.

The transverse displacement w is described within the element by

$$w = N_1 w_1 + N_2 \theta_1 + N_3 w_3 + N_4 \theta_4 \quad (21)$$

where

$$\begin{aligned} N_1 &= 1 - 3x^2/L^2 + 2x^3/L^3 \\ N_2 &= x - 2x^2/L + 2x^3/L^2 \\ N_3 &= -3x^2/L^2 - 2x^3/L^3 \\ N_4 &= -x^2/L + x^3/L^2 \end{aligned} \quad (22)$$

are the third order Hermite polynomials.

For uniformly distributed loads the exact displacement solution is a fourth order polynomial.

Let us suggest the following displacement field

$$w = N_1 w_1 + N_2 \theta_1 + N_3 w_3 + N_4 \theta_4 + N_5 \lambda \quad (23)$$

where $N_5 = x^2(1 - x/L)^2$ and λ is an extra degree of freedom associated with this new element. It can be observed that w_1, θ_1, w_2, θ_2 maintain their physical previous meaning since $N_5|_{x=0} = N_5|_{x=L} = N_5'|_{x=0} = N_5'|_{x=L} = 0$.

The strain energy of the element is given by

$$U = \frac{1}{2} EI \int_0^L \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \quad (24)$$

where E, I are the Young's modulus and the second moment of the cross section, respectively.

If the displacement field (21) is derived twice with respect to x and substituted into expression (24) the element stiffness matrix terms are then obtained.

$$K_{ij}^e = EI \int_0^L N_j'' N_j'' \mathrm{d}x \quad (25)$$

The potential of the loads is given by

$$\Omega_p = \int_0^L Q_y w \, \mathrm{d}x \quad (26)$$

and the nodal equivalent point forces $\mathbf{f}_b^e$ can be obtained

$$f_{b_i}^e = \int_0^L Q_y N_i \mathrm{d}x \quad (27)$$

The element equilibrium equation can now be established for a uniformly distributed load Q_y.

$$\frac{EI}{L^2}\begin{bmatrix} 12/L & 6 & -12/L & 6 & 0 \\ 6 & 4L & -6 & 2L & 0 \\ -12L & -6 & 12/L & -6 & 0 \\ 6 & 2L & -6 & 4L & 0 \\ 0 & 0 & 0 & 0 & 4L^3/5 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ \lambda \end{Bmatrix} = Q_y L \begin{Bmatrix} 1/2 \\ L/12 \\ 1/2 \\ -L/12 \\ L^2/30 \end{Bmatrix} \quad (28)$$

Again λ is not coupled with w_1, θ_1, w_2, θ_2 and is directly calculated, $\lambda = Q_y L^2/24EI$.

Let us consider the case of a cantilever beam with $w_1 = \theta_1 = 0$. The equilibrium equation is reduced to

$$\frac{EI}{L^2}\begin{bmatrix} 12/L & -6 \\ -6 & 44 \end{bmatrix} \begin{Bmatrix} w_2 \\ \theta_2 \end{Bmatrix} = Q_y L \begin{Bmatrix} 1/2 \\ L/12 \end{Bmatrix} \quad (29)$$

yielding the solution

$$w_2 = \frac{Q_y L^4}{8EI}; \quad \theta_2 = \frac{Q_y L^3}{6EI} \quad (30)$$

which is the exact solution for this case.

We can proceed and evaluate the shear forces F and bending moments M, according to the following expressions

$$T = EI \frac{\mathrm{d}^3 w}{\mathrm{d}x^3}; \quad M = EI \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \quad (31)$$

These internal forcs can then be obtained for each node using the matrix equation

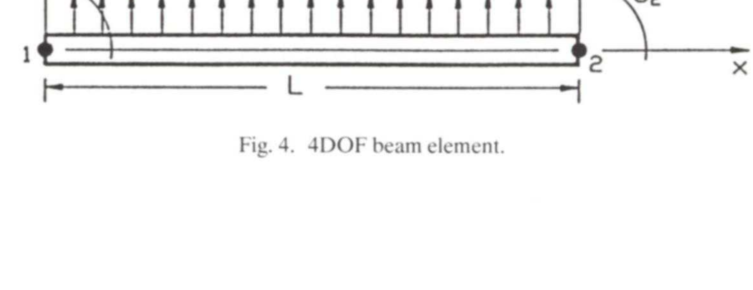

Fig. 4. 4DOF beam element.

$$\begin{Bmatrix} T_1 \\ M_1 \\ T_2 \\ M_2 \end{Bmatrix} = EI \begin{bmatrix} N_1'''|_{x=0} & N_2'''|_{x=0} & N_3'''|_{x=0} & N_4'''|_{x=0} \\ -N_1''|_{x=0} & -N_2''|_{x=0} & -N_3''|_{x=0} & -N_4''|_{x=0} \\ -N_1'''|_{x=L} & -N_2'''|_{x=L} & -N_3'''|_{x=L} & -N_4'''|_{x=L} \\ N_1''|_{x=L} & N_2''|_{x=L} & -N_3''|_{x=L} & -N_4''|_{x=L} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} + EIL \begin{Bmatrix} N_5'''|_{x=0} \\ N_5''|_{x=0} \\ N_5'''|_{x=L} \\ N_5''|_{x=L} \end{Bmatrix} \quad (32)$$

For the case of the cantilever beam referred to above, substituting solution (30) into equation (32)

$$\begin{Bmatrix} T_1 \\ M_1 \\ T_2 \\ M_2 \end{Bmatrix} = Q_y L \begin{Bmatrix} -1/2 \\ -5L/12 \\ 1/2 \\ -L/12 \end{Bmatrix} + Q_y L \begin{Bmatrix} -1/2 \\ -L/12 \\ 1/2 \\ L/12 \end{Bmatrix} = \begin{Bmatrix} -Q_y L \\ -Q_y L^2/2 \\ 0 \\ 0 \end{Bmatrix} \quad (33)$$

which corresponds to the exact solution obtained with the technical theory of beams.

Again it can be seen that expression (32) can be rewritten in the form

$$\mathbf{F}_b^e = \mathbf{K}_b^e \mathbf{q}_b^e - \mathbf{Q}_b^e \quad (34)$$

where $\mathbf{F}_b^e$ are the equilibrium external forces, $\mathbf{K}_b^e$ the element stiffness matrix, $\mathbf{q}_b^e$ the element degrees of freedom, $\mathbf{Q}_b^e$ a correction term which can be directly obtained from the nodal equivalent point forces (27).

Comparing the stiffness terms in (25) with the first matrix on the right-hand side of equation (32) several identities can be established such as

$$\int_0^L N_1'' N_1'' \mathrm{d}x = N_1'''|_{x=0} \quad (35)$$

$$\int_0^L N_2'' N_2'' \mathrm{d}x = N_2'''|_{x=0} \quad (36)$$

$$\int_0^L N_1'' N_2'' \mathrm{d}x = -N_2'''|_{x=0} \quad (37)$$

which can be equally verified, integrating by parts the left-hand side.

CONCLUSIONS

It was shown, within the framework of the finite element formulation, that by introducing additional higher order shape functions and additional new degrees of freedom in the standard 6DOF beam element it is possible to obtain exact solutions for uniformly distributed loaded structures.

These additional shape functions allow the exact evaluation of internal forces by introducing new correcting terms which have been shown to correspond to the nodal equivalent point forces. Although for practical purposes, this result is well known, the formulation presented herein allows a better understanding and provides a formal explanation of the role played by such terms. This method can easily be extended in the development of other elements to model more severe displacement fields (non-uniform distributed loads) without the corresponding increase in the total number of degrees of freedom of the finite element model.

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