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Inviscid Flow at the Trailing Edge of an Airfoil*

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A simple mathematical proof is given that the transit times are the same for two particles, one traveling along the upper surface, the other along the lower surface, of an arbitrary shape airfoil, in an inviscid flow. This result is important in using a common 'intuitive' explanation of the lift phenomenon on such an airfoil.

INTRODUCTION

IN DISCUSSING the two-dimensional (2D), incompressible, inviscid flow around an airfoil, most textbooks in fluid mechanics analyze the structure of the streamlines and velocity field, often using the techniques of complex analysis and conformal mapping (see, for example, [1–3]). What are not often discussed are the fluid particle paths. A good example is that of the paths of two particles that start at the same instant at the front stagnation point of an airfoil, one traveling over the upper surface, the other traveling along the lower surface.

For a symmetric airfoil at zero angle of attack, the two particles will meet at the trailing edge of the airfoil. For an airfoil with camber, or at a non-zero angle of attack, or both, the flow will no longer be symmetric. The question arises as to whether or not the two particles will still meet at the trailing edge. The answer to this question is not obvious, and is important, for the following reason.

It is common in both engineering or science classes in colleges and high schools to explain that a wing with camber and/or an angle of attack generates lift because of the asymmetry in the upper and lower paths, from the front stagnation point to the trailing edge. The concept is that the upper path is longer than the lower path (for positive camber/angle of attack), because the front stagnation point for this asymmetric wing will be somewhere on the lower wing surface. Therefore the particle traveling on the upper path must travel at a higher average speed than the particle traveling on the lower path. Using the Bernoulli principle, this must mean that the average pressure on the upper wing surface must be less than that on the lower surface, generating a net lift. To increase lift, then, one must increase this upper/lower asymmetry—in other words, one must either increase the angle of attack, or increase the camber (or both).

This explanation becomes invalid if the underlying assumption is invalid, namely that the two particles which start at the front stagnation point meet at the trailing edge. In other words, the assumption is that the particle traveling on the upper path travels in the *same period of time* as the particle on the lower path travels.

However, this assumption is not discussed in most texts, including those cited above. Intuitively, it seems a reasonable assumption. One of the few authors to discuss the topic, *Panton* [4], indicates that the particles do *not* meet at the trailing edge, and that the particle traveling on the upper path, though traveling further, arrives there first!

As Panton discusses, one of the reasons that this assumption has not been clearly justified is that it is not trivial, theoretically, to compare the transit times of the two particles. In theory, both transit times are infinite because they both leave from a stagnation point. Panton avoids this difficulty by considering two particles starting at an infinitesimal distance ε above and below the front stagnation point, and then computes their transit times, then letting $\varepsilon \to 0$. With this method, it seems, as mentioned above, that the assumption is unfounded.

In the following analysis, a comparison of the transit times for an 'upper' and 'lower' particle will be made, using a different technique than Panton's. To simplify the analysis, instead of an arbitrary airfoil with camber and/or an angle of attack, flow over a circular cylinder with circulation will be studied. As is well known [3], by a suitable conformal transformation, such a flow can be transformed into any desired airfoil shape. Hence, results obtained from the simpler geometry will be applicable to the arbitrary shape airfoil.

For a circular cylinder with radius a and circulation Γ in a flow with free stream velocity U, the velocity field is given by [4]:

$$v_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \qquad (1)$$

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$$v_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r} \tag{2}$$

where (r, θ) are the polar coordinates, with (v_r, v_θ) being the corresponding velocity components (see Fig. 1).

On the surface of the cylinder $r = a$, so

$$v_r = 0 \tag{3}$$

$$v_\theta = -2U\sin\theta - \frac{\Gamma}{2\pi r} \tag{4}$$

From Eq. (4), the two stagnation points, θ_1 and θ_2, are given by

$$\theta = \sin^{-1}\left(-\frac{\Gamma}{4\pi a U}\right) \tag{5}$$

For a particle at stagnation point θ_1 and traveling along the upper surface, the time to travel to θ_2 will be

$$t_u = \int_{\theta_{1_{c_u}}}^{\theta_2} \frac{a\,d\theta}{v_\theta} \tag{6}$$

where c_u denotes the upper path. In Eq. (6) $d\theta$ and v_θ are both negative, yielding a positive transit time.

For a particle starting at θ_1, following the lower path c_l to θ_2, the transit time is

$$t_l = \int_{\theta_{1_{c_l}}}^{\theta_2} \frac{a\,d\theta}{v_\theta} \tag{7}$$

where t_l is again a positive time.

Equations (6) and (7) will be used to test the assumption under question. Note that both t_u and t_l go to infinity, but it is assumed that they are equal: if this assumption turns out to be invalid, a contradiction will appear in the following deduction.

Hence, assume $t_u = t_l$, or

$$\int_{\theta_{1_{c_u}}}^{\theta_2} \frac{a\,d\theta}{v_\theta} = \int_{\theta_{1_{c_l}}}^{\theta_2} \frac{a\,d\theta}{v_\theta} \tag{8}$$

Then

$$\int_{\theta_{1_{c_u}}}^{\theta_2} \frac{a\,d\theta}{v_\theta} + \int_{\theta_{2_{c_l}}}^{\theta_1} \frac{a\,d\theta}{v_\theta} = 0 \tag{9}$$

or

$$\oint \frac{d\theta}{v_\theta} = 0 \tag{10}$$

Using Eq. (4) in Eq. (10) yields

$$-\oint \frac{d\theta}{\left(2U\sin\theta + \frac{\Gamma}{2\pi a}\right)} = 0 \tag{11}$$

or

$$\oint \frac{d\theta}{(\sin\theta + \lambda)} = 0 \tag{12}$$

where

$$\lambda = \frac{\Gamma}{4\pi a U} < 1 \tag{13}$$

Equation (13) ensures that there are two stagnation points [3]. For $\lambda = 1$ there would be one stagnation point; for $\lambda > 1$ there are no stagnation points on the cylinder, and a circulatory flow exists

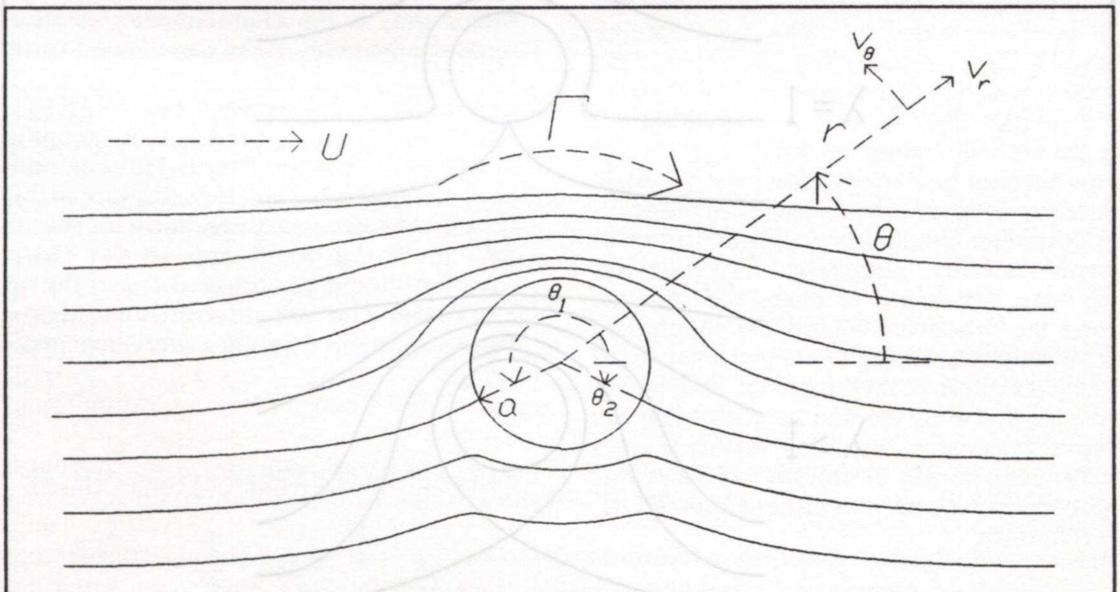

Fig. 1 Uniform flow over a cylinder with circulation.

around the cylinder. These conditions are illustrated in Fig. 2.

To integrate Eq. (12), without loss of generality one can change variables:

$$z = e^{i\theta} \qquad (14)$$

Hence

$$dz = iz\,d\theta \qquad (15)$$

and

$$\sin\theta = \frac{i(z^2-1)}{2z} \qquad (16)$$

Substituting Eqs (15) and (16) in Eq. (12) yields

$$\oint \frac{dz}{z\left(-\frac{i(z^2-1)}{2z}+\lambda\right)} = 0 \qquad (17)$$

or

$$\oint \frac{dz}{(z^2+i2\lambda z-1)} = \oint \frac{dz}{(z-z_1)(z-z_2)} = 0 \qquad (18)$$

where

$$z_1 = -i\lambda - \sqrt{1-\lambda^2} \qquad (19)$$

$$z_1 = -i\lambda + \sqrt{1-\lambda^2} \qquad (20)$$

corresponding to the stagnation points at θ_1 and θ_2 respectively. The integral in Eq. (18) has two simple poles, one at z_1 and one at z_2, both on the perimenter, and hence [5] the contour integral is:

$$\oint \frac{dz}{(z-z_1)(z-z_2)} = 2\pi i \sum \text{Res(interior space)} + \pi i \sum \text{Res(perimeter)}$$

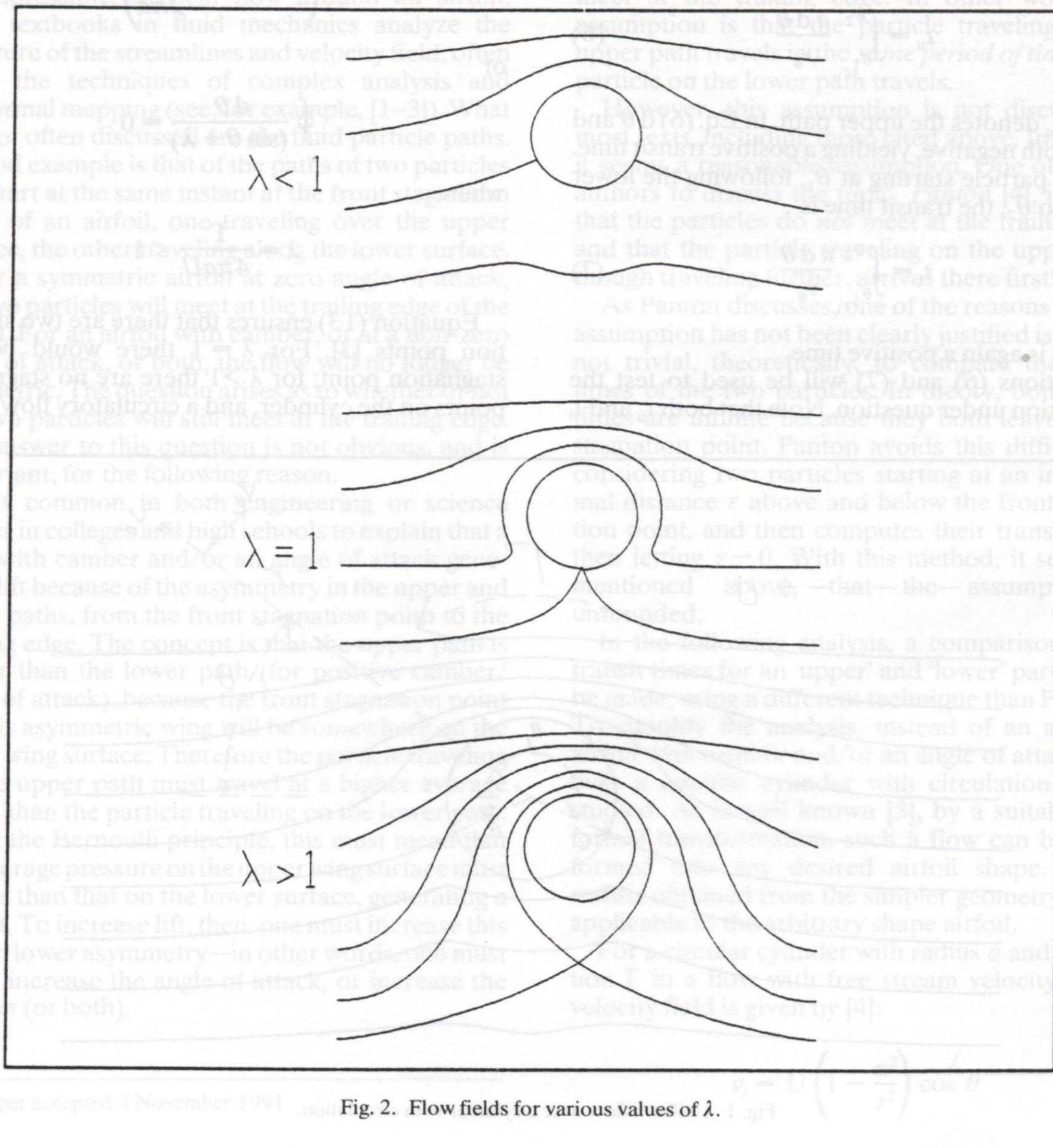

Fig. 2. Flow fields for various values of λ.

$$= 2\pi i(0) + \pi i(\text{Res}(z_1) + \text{Res}(z_2))$$

$$= \pi i\left(\frac{1}{(z_1 - z_2) + (z_2 - z_1)}\right)$$

$$= 0 \qquad (21)$$

which is consistent with Eq. (17). Thus, the validity of Eq. (8) is demonstrated, and it has been proved that the two particles, starting at the front stagnation point, do in fact meet again at the rear stagnation point. This result, as mentioned above, is valid for cambered airfoils at an angle of attack, as well as the cylinder with circulation.

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