Int. J. Appl. Engng Ed. Vol. 7, No. 5, pp. 381–387, 1991
Printed in Great Britain.

0742–0269/91 \$3.00+0.00

Integrated View of Methods in Buckling Analysis*

C. M. WANG
Department of Civil Engineering, National University of Singapore, Singapore
S. KITIPORNCHAI†
Department of Civil Engineering, University of Queensland, St Lucia, Queensland, Australia

An integrated view of numerical techniques for determining the elastic buckling load of structural members is presented. The commonly used techniques can be grouped under two main approaches, namely the static–kinematic approach and the work–energy approach. Each technique is described briefly and its advantages highlighted. More importantly, the connecting paths between the various techniques are discussed. An understanding of the integrated view and the techniques will provide a greater flexibility for solving buckling problems by employing the most expedient techniques for solution. The various methods may also be used independently to solve the same problem to confirm the accuracy, validity and convergence of the solutions.

Notation

c_i	coefficients of approximated displacement functions.
$\{\mathbf{d}\}$	element buckling nodal displacement vector
$g(x)$	assumed functions
$h(x)$	constraints
$[\mathbf{k}_e]$	element stiffness matrix
$[\mathbf{K}]$	assembled global stiffness matrix
n	number of chosen linear combination of functions
$[\mathbf{T}]$	transformation matrix
U	strain energy
U_e	strain energy of an element
V	potential energy of loads
V_e	potential energy of loads of an element
W	work done by loads
W^*	work done per unit load
y	displacement function
z	approximated displacement function
$\{\mathbf{\Delta}\}$	assembled nodal displacement vector
ε	error
λ	buckling load
Λ	Lagrangian multiplier
Π_C	complementary energy functional
Π_e	potential energy of an element
Π_P	potential energy functional
Π_R	Hellinger–Reissner functional
Π_W	Hu–Washizu functional

INTRODUCTION

BUCKLING is an important consideration in the design of steel structures. Elastic buckling is the most fundamental form, and the study of elastic buckling is an essential step towards understanding the structural stability behaviour of complex structures and the analysis of structures incorporating the more complicated inelastic buckling. The load at which elastic buckling occurs is also important, because it provides the upper limit to the member buckling strength and is commonly used in design codes as the basis for which the ultimate design capacity of the members is derived.

In open literature and standard texts, buckling loads for different kinds of structures under various loading and boundary conditions are often expressed using approximate simple formulae and design charts to aid designers in estimating the buckling strength of structural members. It is still necessary, however, for designers to perform the buckling analysis if more accurate results are required or if there is no standard solution available.

Apart from a few problems (such as the elastic buckling of perfect and prismatic struts under an axial force or the lateral buckling of simply supported beams under uniform moment and axial force), it is generally rather laborious and in some cases impossible to obtain exact analytical solutions. Thus, it becomes necessary to resort to numerical techniques. There are a large number of techniques available which can be grouped under two general approaches as shown in the chart of Fig. 1. The two approaches, *viz.* the static–kinematic approach and the work–energy approach, correspond to the different strategies used in satisfying the state of neutral equilibrium for the deformed member.

* Paper accepted 29 October 1991.
† Dr. Kitipornchai is Associate Professor

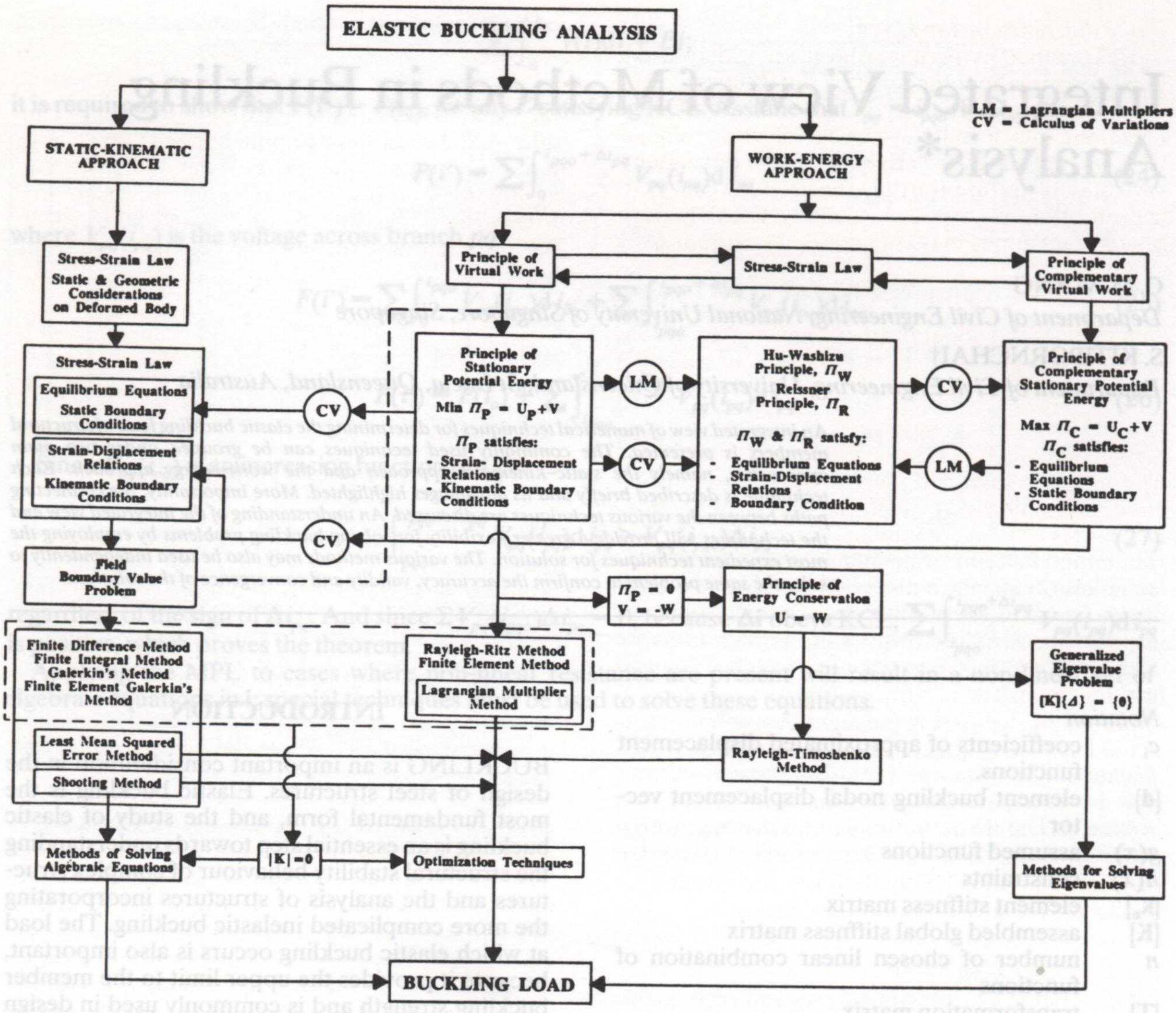

Fig. 1 Relationships of methods in buckling analysis.

Even though there are many ways of determining the buckling load, in practice designers tend to use only a small number of familiar techniques. The objective of this paper is to collate all the different common techniques and present them in an overall integrated view so that designers may (a) have a global view and appreciate the connections between the techniques, (b) possess greater flexibility when solving various kinds of problems, (c) exploit the various advantages associated with each method and (d) use independent methods to check the accuracy, validity and convergence of the solutions.

STATIC–KINEMATIC APPROACH

In this approach, the field differential equations are derived from (a) the assumed stress–strain law, (b) equilibrium equations and (c) strain–displacement relations. The last two conditions are obtained from static and geometric considerations on a free body of the deformed member (or the slightly bent form). Together with the given boundary conditions, the resulting two-point boundary value problems are to be solved for the buckling load. In some rare cases, closed form analytical solutions are possible and if the differential equations are in the forms of either Bessel's equation or Legendre's equation, the power series method [1] may be used to derive semi-analytical solutions. In general, however, it is more convenient and often necessary to employ numerical techniques. There are several common techniques, including the finite difference method, the finite integral method, the least mean squared error method, Galerkin's method, the finite element Galerkin method and the shooting method. These are briefly reviewed below. Other less commonly used methods can be found in the books by Keller [2] and Na [3].

Finite difference method [1, 4]

This method is perhaps the most widely used numerical method for solving differential equations. The method converts the set of differential equations into a finite system of algebraic equations by replacing the derivatives of the dependent variables using appropriate finite differences at chosen nodal points. The resulting linear homogeneous equations may collectively be written as

$$[\mathbf{K}]\{\mathbf{\Delta}\} = \{0\} \quad (1)$$

where $\{\mathbf{\Delta}\}$ is the nodal displacement vector. For a nontrivial solution, the determinant of the $[\mathbf{K}]$ matrix must vanish, i.e.

$$|\mathbf{K}| = 0 \quad (2)$$

The buckling load can then be determined using either Equation (1) with matrix manipulation algorithms [5] that compute the eigenvalues or Equation (2) with root finding algorithms [1]. Equation (2) may also be casted into a univariable optimization problem with the objective function taken as the absolute of the determinant (that is, ABS(det[**K**])) and the decision variable is the buckling load. The accuracy of the method can be improved by decreasing the interval size but this is at the expense of greater computational effort. A quicker and simpler way of improving the accuracy is by using Richardson's extrapolation scheme [6].

A useful feature of the finite difference method is that it can be used to solve partial differential equations and thus the method is not restricted to solving only one-dimensional structures.

Finite integral method [7]

This method, in effect, uses an opposite technique to the finite difference method. Instead of using a finite difference representation of the differential operators, the differential equation is first transformed into an integral equation in the highest derivative of the dependent variable. The integral equation is then replaced by a finite set of algebraic equations using finite integral representations of the integral operators. The algebraic equations are then solved using the same techniques employed in the finite difference method.

The finite integral method gives a more accurate solution than the finite difference method [7]. This is because of the superiority of the numerical integration process over the differentiation process on which the finite difference is based. The superiority increases when the interval size is decreased.

Least mean squared error method [8]

Unlike the finite difference and the finite integral methods which select some discrete points, the least mean squared error method considers all points of the range x_0 to x_f. After selecting a linear combination of functions, $g(x)$, which satisfy all the boundary conditions, to approximate the displacement function (or dependent variable), y, i.e.

$$y \simeq z = \sum_{i=1}^{n} c_i g(x) \quad (3)$$

the method seeks an approximate solution to the differential equation by minimizing the error, ε, in the differential equation corresponding to the approximation. If ε were zero for all values of x within the range, the solution would be exact. In buckling analyses, the following error functional is minimized.

$$\min_{\lambda, c_i} F = \int_{x_0}^{x_f} \varepsilon^2 \mathrm{d}x \quad (4)$$

where λ is the buckling load and c_i are the coefficients of the approximate displacement function. The minimization process can be performed using standard optimization techniques [9].

Galerkin method [8]

Galerkin's method is similar to the least mean squared error method except that it minimizes the error, ε, by choosing the n coefficients in the function in such a manner that n distinct weighted means of the error, taken throughout the range of representation shall be zero, i.e.

$$\int_{x_0}^{x_f} \varepsilon \frac{\partial z}{\partial c_i} \mathrm{d}x = 0, \qquad i = 1, 2, \ldots, n \quad (5)$$

where z is the approximated dependent variable given by Equation (3). Equation (5) yields a set of algebraic equations which can be written in the form of Equation (1) and the buckling load can then be obtained by solving the standard eigenvalue problem or by solving for the zero of the determinant when the problem is expressed as in Equation (2). In stability applications, ε can be interpreted as a generalized force, and the multipliers used to weight the errors are the virtual displacements corresponding to the increments of each of the generalized coordinates in turn. Thus the vanishing of the weighted mean is interpreted as the vanishing of the virtual work in the appropriate displacement. The degree of accuracy attained can be increased indefinitely by increasing the number of independent functions employed, but this entails a great increase in computational effort. When the functions are well chosen, however, an excellent approximation can be obtained with a very small value of n.

If the trial functions are piecewise polynomials instead of linear combinations of functions, then the method is called the finite element Galerkin's method [10]. Obviously, this method becomes more powerful and can handle complicated displacement functions.

Shooting method [2, 3]

This method of solving the two-point boundary value problem first converts the Nth order differential equation into a set of first order differential equations. The first order differential equations are then integrated forward using the Runge–Kutta algorithm [1] with some assumed values for the unknown initial values and the load parameter. The step size for the integration must be sufficiently small for accurate solution. The terminal boundary conditions are then to be satisfied by systematically adjusting these unknown parameters. The problem reduces to solving a few algebraic equations which express the terminal boundary conditions [11]. Alternatively, the terminal boundary conditions may be satisfied by minimizing the objective function consisting of the sum of the L_1 norm of the error ε with respect to the unknown parameters using any standard optimization technique [12].

Unlike other methods such as the finite difference, finite integral, Galerkin's method, Rayleigh–Ritz, and finite element method, the shooting method avoids determining the zero of the characteristic determinant (Equation (2)). The method therefore requires neither matrix manipulation nor inversion to determine the buckling load and mode shape (eigenfunction), making the method programmable in a small personal computer because computational effort and storage space are reduced. In addition, the method generates the eigenfunction simultaneously with the eigenvalue through the solution of the differential equation while the other methods require a separate process. The eigenvalues may be generated one at a time and their order determined from the eigenfunction, without wasting computing time to evaluate all the eigenvalues of large matrices. Its usefulness is enhanced when solving large deflection and postbuckling problems, which are described by highly nonlinear differential equations. Finally, because there is no need for discretization, it can compute the eigenvalues and eigenfunctions with greater accuracy, particularly when the cross-sectional area of the structural member is varying. However, this method is restricted to problems which are defined by ordinary differential equations.

WORK–ENERGY APPROACH [13, 18]

Unlike the static–kinematic approach, where the equilibrium of the member is established by requiring the sum of the generalized forces to vanish, this approach satisfies the equilibrium condition of the deformed member via the virtual work concept. Referring to Fig. 1, this approach can be based on either the principle of virtual work or the principle of complementary virtual work, which are related to each other through the stress–strain law. In elastic buckling problems, the two general principles may be reduced to the principle of stationary potential energy and the principle of stationary complementary potential energy, respectively. For the former case, this can be done by establishing the potential energy functional, Π_P, and assuming the external forces to be unchanged during displacement variation. For the latter case, this can be done by establishing the complementary energy functional Π_C, and assuming the kinematic boundary conditions to be unchanged during stress variation.

Using Lagrangian multipliers, the energy functionals Π_P, Π_C may be generalized to other forms of functional such as the Hu–Washizu Π_W functional or the Hellinger–Reissner functional Π_R. The buckling load can be determined by minimizing the functionals. For the functional: (a) Π_P, the strain–displacement relations and the kinematic boundary conditions are satisfied at the outset and the equilibrium equations and static boundary conditions are satisfied indirectly by the stationarity conditions; (b) Π_C the equilibrium equations and the static boundary conditions are satisfied at the outset while the strain–displacement relations and the kinematic boundary conditions are satisfied indirectly by the stationarity conditions; and (c) Π_W, Π_R, the equilibrium equations, strain–displacement relations and the boundary conditions must be satisfied at the outset.

Since the usual preferred avenue of the energy approach is to use the principle of virtual work for inelastic problems or the principle of stationary potential energy for elastic problems, the methods associated with their complementary counterparts will not be discussed. For details, the reader is referred to references [16–18].

Another important principle which applies for elastic problems is the principle of energy conservation. This principle may be also derived from the principle of stationary potential energy since the minimum value of Π_P is zero and the potential energy of the loads is equal to the negative value of the work done by the loads. On the basis of the principle of energy conservation and the principle of stationary potential energy, various energy methods were proposed by early researchers for determining the buckling load and these methods are briefly reviewed below.

Rayleigh–Timoshenko method [14]

The Rayleigh–Timoshenko method is based on the conservation of energy principle. When the member buckles under the applied loads, the strain energy stored is given by U. The external work, W, is done at the same time as a result of the movement of the loads. For energy conservation

$$U = W \tag{6}$$

The load parameter, λ, may be factored out from the work expression yielding the Rayleigh quotient. The buckling load is then obtained by minimizing the quotient with respect to the displacement function, y, i.e.

$$\min_y \lambda = \frac{U}{W^*} \tag{7}$$

where W^* is the work done per unit load. If the exact expression for the displacement function, y, is known, the critical load can be obtained directly from the quotient. Otherwise, the unknown infinite dimensional displacement function may be parameterized by spline functions, or trigonometric series, or polynomials that satisfy the boundary conditions. The quotient can then be minimized with respect to the displacement coefficients. It is necessary to normalize the displacement function so as to avoid a trivial solution. It can be shown from the Rayleigh quotient and the orthogonality relationships that the energy result is always an upper bound to the true one. A convergence study therefore must be carried out to establish the convergence of the solution to the required accuracy.

Rayleigh–Ritz method [15]

This method uses the principle of stationary potential energy. The principle follows from the more general principle of virtual work which states that a body is in equilibrium if the total virtual work done by all the forces (both internal and external) acting on the body for any arbitrary virtual (or fictitious) displacement is zero. It implies that neutral equilibrium corresponds to a minimum of the total potential energy, Π, i.e.

$$\partial\Pi = \partial(U + V) = 0 \qquad (8)$$

where ∂ represents the change in quantity caused by a virtual displacement, U the strain energy and V the potential energy of the loads which is the negative value of the work done. To solve for the buckling load, the displacement function is approximated by some linear combinations of functions as given in Equation (3). The functions need only satisfy the kinematic boundary conditions; otherwise erroneous solutions are obtained. Then the Rayleigh–Ritz method requires

$$\frac{\partial\Pi}{\partial c_i} = 0, \qquad i = 1, 2, \ldots, n \qquad (9)$$

Equation (9) generates a system of linear algebraic homogeneous equations of the form given in Equation (1). Being an energy method, the solution obtained is an upper bound one when compared to the exact solution. A convergence study is necessary to determine the accuracy of the upper bound solution.

In view of the stress–strain law, the strain energy, U, may be expressed in an alternative form of generalized stress resultants instead of generalized strains. This functional is termed the complementary total potential energy functional [16]. Using this formulation, the generalized stress resultants are approximated by a series which must satisfy the static boundary conditions. Taking the stationarity conditions of the complementary energy functional with respect to the coefficients of the approximated generalized stress resultants yields a set of homogeneous linear equations which can be written as in Equation (1). The buckling load is then obtained by solving the generalized eigenvalue problem.

Finite element method [5]

The finite element method is an extension of the Rayleigh–Ritz method in that it involves the division of the structural member into a number of elements. The element stiffness matrix $[\mathbf{k}_e]$ can then be obtained from the total potential energy of the element

$$\Pi_e = (U_e + V_e) = \tfrac{1}{2}\{\mathbf{d}\}^T[\mathbf{k}_e]\{\mathbf{d}\} \qquad (10)$$

in which $\{\mathbf{d}\}$ is the element buckling nodal displacements vector. From the compatibility relation,

$$\{\mathbf{d}\} = [\mathbf{T}]\{\mathbf{\Delta}\} \qquad (11)$$

in which $\{\mathbf{\Delta}\}$ are the structure joint displacements and $[\mathbf{T}]$ the transformation matrix. In view of Equation (11), Equation (10) may be written as

$$\Pi_e = \tfrac{1}{2}\{\mathbf{\Delta}\}^T[\mathbf{K}_e]\{\mathbf{\Delta}\} \qquad (12)$$

in which $[\mathbf{K}_e] = [\mathbf{T}]^T[\mathbf{k}_e][\mathbf{T}]$. Repeating this for all the elements in the structure and summing up yields

$$\Pi = \sum \Pi_e = \tfrac{1}{2}\{\mathbf{\Delta}\}^T[\mathbf{K}_T]\{\mathbf{\Delta}\} \qquad (13)$$

in which $[\mathbf{K}_T] = \Sigma[\mathbf{K}_e]$.

During buckling from the straight position to an adjacent out-of-plane equilibrium position, the energy is conserved, thus the total energy change is zero, implying

$$\Pi = \tfrac{1}{2}\{\mathbf{\Delta}\}^T[\mathbf{K}_T]\{\mathbf{\Delta}\} = 0 \qquad (14)$$

which is similar to Equation (1) and the buckling load thus can be determined using one of the eigenvalue methods. Alternatively, since the displacement vector $\{\mathbf{\Delta}\}$ is nonzero for the adjacent equilibrium position, Equation (14) implies Equation (2).

In view of the piecewise polynomial approximations to the dependent variable (or displacement functions), the finite element method has some important advantages over the Rayleigh–Ritz method. The former method can easily handle the rapidly changing shape of displacement functions and need not be bothered with the selection of appropriate Ritz functions. In some cases where the few chosen Ritz functions closely approximate the exact displacement functions, however, very neat and compact formulae may be obtained which would be quite difficult to develop using the finite element method because many results are needed to perform an adequate curve-fitting exercise. Even then, the formulae may not be as accurate and compact as the ones obtained using the Rayleigh–Ritz method.

When dealing with two- or three-dimensional elongated structural members, the number of degrees of freedom may be reduced by having the finite strip method [19] instead. The finite strip method may be viewed as a special case of the finite element method with elongated elements whose displacement function for the longer side is approximated by trigonometric series instead of polynomials.

Lagrangian multiplier method [20–23]

This method is a variation of the Rayleigh–Ritz method. Instead of finding the Ritz functions that satisfy the kinematic boundary conditions, the method requires the approximated function, z, to be constrained so that the geometric boundary conditions are satisfied. Let these constraining relationships be denoted by

$$\sum_{j=1}^{m} h_j(c_1, c_2, \ldots, c_n) = 0 \qquad (15)$$

where m is the total number of kinematic boundary conditions. Using the Lagrangian method, the augmented total potential energy, Π^* becomes

$$\Pi^* = \Pi + \sum_{j=1}^{m} \Lambda_j h_j \qquad (16)$$

where Λ_j are undetermined constants called Lagrangian multipliers. For stationarity,

$$\frac{\partial \Pi^*}{\partial c_i} = 0; \quad i = 1, 2, \ldots, n$$
$$\frac{\partial \Pi^*}{\partial \Lambda_j} = 0; \quad j = 1, 2, \ldots, m \qquad (17)$$

Equation (17) yields a set of $(n+m)$ homogeneous linear equations which can be written as in Equation (1) and the buckling load computed by solving for the vanishing of the determinant $|\mathbf{K}|$. The buckling load can also be determined by solving the standard constrained optimization problem where the objective function is Π and the constraints given by Equation (15) via any standard mathematical programming methods such as the efficient Schittkowski's algorithm [24].

Remarks

To obtain the critical load, it can be seen that all the methods ultimately lead to (a) solving a set of algebraic equations using algorithms such as that of Newton [1] or Broyden linear search [25]; (b) minimizing an objective function/functional using optimization techniques [9]; or (c) solving a generalized eigenvalue problem via algorithms such as vector iteration method [5], transformation method [5], polynomial iteration techniques [5], Sturm sequence property methods [5] or Wittrick and William method [26].

RELATIONSHIP BETWEEN THE TWO APPROACHES

From Fig. 1, it can be seen that there are relations between the two lines of approach. By forming the total potential energy in the work-energy approach and then minimizing with respect to the displacement functions using calculus of variations, the equilibrium differential equations and the natural boundary conditions can be derived [18]. This strategy of deriving the equilibrium differential equations is becoming more popular because it is rather easier when compared to the conventional approach of analysing a free body of the deformed member and obtaining the equilibrium equations from statical and geometrical considerations. On the basis of this approach, Papangelis and Trahair [27] have recently discovered errors in the governing differential equations derived previously from the static–kinematic approach for the flexural–torsional buckling of circular arches.

Unfortunately, the reverse process of developing an energy functional from a set of differential equations is not so straightforward. Either the functional is obtained by guesswork or under certain conditions, the principle of virtual work or the Galerkin's method may be used. In some cases, this reverse process is not possible as an energy functional may not exist for the differential equation. An example is the governing differential equation for the equilibrium of shear forces for multistorey shear-wall frame structures where the wall and the frames are elastically supported at the base [28].

The various work–energy principles are also interrelated. The principle of virtual work/stationary potential energy and the one based on the complementary virtual work/stationary complementary potential energy are related through the stress–strain law. Moreover, it can be shown that the minimum value of the total potential energy is zero which implies the principle of energy conservation.

CONCLUDING REMARKS

An integrated view of common numerical techniques for stability analysis is presented. Although the discussion concerns elastic buckling, all of the methods under the static–kinematic approach can be applied to inelastic buckling or any other engineering problems involving the solving of differential equations. When using the work–energy approach for tackling inelastic buckling problems, the more general virtual work theorems have to be used instead of the principles of potential energy and principle of energy conservation. The usual technique used in conjunction with the principle of virtual work is the finite element method. For large displacement problems, the differential equations are nonlinear. Thus, some of the listed methods become inefficient. Such probems are best solved using the shooting method or the finite element method.

Acknowledgements—The paper was written while the first author was on sabbatical leave in the Department of Civil Engineering, The University of Queensland, Australia. The authors wish to thank Mr Warren H. Traves of Gutteridge Haskins and Davey Pty Ltd for proof-reading the manuscript.

REFERENCES

1. E. Kreyszig, *Advanced Engineering Mathematics*, 5th ed. John Wiley and Sons, New York (1983).
2. H. B. Keller, *Numerical Methods for Two-point Boundary Value Problems*. Blaisdell, New York (1968).
3. Y. T. Na, *Computational Methods in Engineering Boundary Value Problems*. Academic Press, New York (1979).
4. M. G. Salvadori and M. L. Baron, *Numerical Methods in Engineering*, 2nd edn. Prentice Hall, Englewood Cliffs, NJ (1961).
5. K. J. Bathe and E. L. Wilson, *Numerical Methods in Finite Element Analysis*. Prentice Hall, Englewood Cliffs, NJ (1976).
6. M. G. Salvadori, Numerical computation of buckling loads by finite differences. *Trans. ASCE*, **116** (1951).
7. P. T. Brown and N. S. Trahair, Finite integral solution of differential equations. *Civil Engng Trans.*, Inst. Engineers, Australia **CE10**, No. 2, 193–196 (1968).
8. W. J. Duncan, Galerkin's Method in Mechanics and Differential Equations. Technical Report of the Aeronautical Research Committee, Reports and Memoranda No. 1798, pp. 484–516 (1937).
9. G. V. Reklaitis, A. Ravindran and K. M. Ragsdell, *Engineering Optimization: Methods and Applications*. John Wiley and Sons, New York (1983).
10. G. Fairweather, *Finite Element Galerkin Methods for Differential Equations*. M. Dekker, New York (1978).
11. C. J. Goh and C. M. Wang, Generalized shooting method for elastic stability analysis and optimization of structural members. *Computers and Structures*, **38**, No. 1, 73–81 (1991).
12. C. M. Wang and S. Kitipornchai, Shooting-optimization technique for large deflection analysis. *Engineering Structures*, **14**, (1992).
13. H. L. Langhaar, *Energy Methods in Applied Mechanics*. John Wiley, New York (1962).
14. S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability*. McGraw-Hill, New York (1961).
15. W. Ritz, Uber eine neue methode zur losung gewisser varionsprobleme der mathematischen physik. *Z. reine Angew. Math.*, **135**, 1–61 (1909).
16. C. Libove, Elastic stability. *Handbook of Engineering Mechanics* (Flugge, Editor), Chap. 44. McGraw-Hill, New York (1962).
17. E. Reissner, on a variational theorem for finite elastic deformations. *J. Math. Phys.*, **32**, 129–135 (1953).
18. K. Washizu, *Variational Methods in Elasticity and Plasticity*, Pergamon Press, New York (1967), 2nd ed. (1975).
19. Y. K. Cheung, *Finite Strip Method in Structural Analysis*. Pergamon, Oxford (1976).
20. E. Trefftz., Die bestimmung der knicklast gedrucker rechteckiger platten, *Z. Angew Math. Mech.*, **15**, 339–348 (1935).
21. E. Trefftz, Ein gegenstuck zum ritzchen verfahren, , *Proc. 2nd International Congress of Applied Mechanics*, Zurich, Switzerland, pp. 131–137 (1926).
22. B. Budiansky and P. C. Hu, The Lagrangian Multiplier Method for Finding Upper and Lower Limits to Critical Stresses of Clamped Plates. NACA Report No. 848 (1946).
23. B. Budiansky, P. C. Hu and R. W. Connor, Notes on the Lagrangian Multiplier Method in Elastic-stability Analysis. NACA Technical Note No. 1558 (1948).
24. K. Schittkowski, NLPQL—a Fortran subroutine for solving constrained nonlinear progamming problems. *Operation Res. Ann.*, **5**, 485–500 (1985).
25. C. G. Broyden, A class of methods for solving nonlinear simultaneous equations. *Math. Computer*, **19**, 577–593 (1965).
26. W. H. Wittrick and F. W. Williams, A general algorithm for computing natural frequencies of elastic structures. *Mech. Appl. Math.*, **XXIV**, Part 3, 263–284 (1971).
27. J. P. Papangelis and N. S. Trahair, Flexural–torsional buckling of arches. *J. Struc. Engng*, ASCE, **112** (11), 2494–2511 (November 1986).
28. C. M. Wang, K. K. Ang and S. T. Quek, Stability formulas for shear-wall frame structures. *Building Environ. J.*, **26**, No. 2, 217–222 (1991).