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An Alternative Approach for Teaching Circuit Analysis*

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Kirchhoff's Current Law (KCL) and the Minimum Power Law (MPL) are used for analysis of circuits that do not contain dependent sources, without relying on Kirchhoff's Voltage law (KVL) or Ohm's Law (OL). The result is a set of algebraic equations, the solution of which gives the branch currents and nodal voltages. The method is generalized to resolve circuits containing dependent and independent sources.

INTRODUCTION

ABOUT a century ago, James Clerk Maxwell wrote:

> In any system of conductors in which there are no internal electromotive forces the heat generated by currents distributed in accordance with Ohm's Law is less than if the currents had been distributed in any other manner consistent with the actual conditions of supply and outflow of current. . . . We can prove in a similar way that when there are electromotive forces in the different branches the currents adjust themselves so that $\Sigma RC^2 - 2\Sigma EC$ is a minimum, where E is the electromotive force in the branch when the current is C . . . This is often the most convenient way of finding the distribution of current among the conductors [1].

Surprisingly, the above-mentioned theorems are rarely discussed, let alone used, in textbooks on circuit theory—despite the ease and elegance of using them and despite the important conclusions relating to power dissipation in this age of energy awareness.

A course in circuit analysis is one of the most important courses in the electrical engineering curriculum and it is increasingly becoming common for students of other engineering disciplines. Usually analysis techniques are developed based on Ohm's Law (OL) and Kirchhoff's Voltage and Current Laws (KVL and KCL), and these are accepted as axiomatic. KCL can be easily accepted as axiomatic because of the many examples around us that can be used to illustrate it. KVL, however, is not as easily acceptable initially, especially to non-electrical engineering students. Based on these laws, methods for circuit analysis, like the mesh method (MM) and the nodal method (NM), were developed; each of those methods has its own limitations. In particular, neither MM nor NM can directly handle all types of dependent sources.

An alternative introductory approach for teaching circuit theory is to use KCL and the Minimum Power Law (MPL) as axiomatic. A power function, F, is defined and it will be shown that branch currents are such that this power function is minimal subject to KCL. Dependent sources will be treated as independent sources when minimizing F, and additional equations are added to describe the dependency of those sources. The minimization process results in an OL equation for every branch current and a KCL equation for every unknown nodal voltage. The necessary circuit theory tools are only KCL and the MPL to be introduced. Writing the power function F is easy and straightforward. The mental effort and the possibility of errors are reduced when using this approach. The computer is used as a computational tool to solve equations.

When comparing the methods developed in this paper with the MM and the NM, it will be apparent that the limitations of those methods are overcome. As the unknowns in the resulting set of equations are the branch currents and the unknown nodal voltages, the resulting set of equations to be solved is larger than when applying the MM or the NM. This apparent disadvantage is insignificant considering the fact that the skill required to write the equations to be solved will be quickly mastered, and most engineering students have programs to solve equations at their fingertips. Also, it will be found that the matrix of coefficients is sparse, making it easy to solve the resulting equations without a computer.

It is emphasized here that the purpose of this paper is not to replace existing methods of circuit analysis; these methods took decades to develop and refine. The presented ideas, however, can be taught in conjunction with existing methods and

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can be the nucleus of major development in this area.

In what follows, the theorems will be presented, followed by special cases of the MPL and then the general case. The techniques presented can be applied to AC circuit analysis, by substituting the usual phasor or complex quantities in place of the real variables and parameters used herein.

THE MINIMUM POWER LAW

THEOREM 1 *In a DC circuit having no voltage sources the injected currents will be distributed in the branches such that the power loss is minimal, subject to KCL.*

The proof of this theorem appears in Appendix I, but it is not necessary to introduce the proof to the students, because the suggested alternative approach is to accept this theorem as axiomatic and use it together with KCL to resolve circuits without referring to OL or KVL.

Application

The problem of analyzing a circuit that has independent current sources is reduced to minimizing a function of the branch currents F, where $F = \Sigma i_{pq}^2 R_{pq}$. F will be called the power function. The minimization is subject to boundary conditions, $A \cdot i = I$, which are KCL at the nodes, and A is a connection matrix. The only circuit theory background needed is KCL and the MPL theorem. Lagrange multipliers or other techniques can be used to minimize F. The minimization process will result in a set of linear equations that are OL for every branch and KCL for every independent node. The equations can be written directly and with ease as the following example illustrates.

Example 1

Consider a circuit having four nodes including the reference node, one current source at node 2, I_2, and four branches as shown in Fig. 1.

The power function F to be minimized is:

$$F = R_1 i_1^2 + R_2 i_2^2 + R_3 i_3^2 + R_4 i_4^2 \tag{1}$$

subject to equality constraints:

$$-i_1 - i_2 = 0 \tag{2}$$

$$I_2 + i_2 - i_3\ 3\ 0 \tag{3}$$

$$i_1 + i_3 - i_4 = 0 \tag{4}$$

To apply the method of Lagrange multipliers, a function Y is formed from Eqs (1) and (2)–(4):

$$Y = R_1 i_1^2 + R_2 i_2^2 + R_3 i_3^2 + R_4 i_4^2 + V_1(-i_1 - i_2) + V_2(I_2 + i_2 - i_3) + V_3(i_1 + i_3 - i_4) \tag{5}$$

where the Vs are Lagrange multipliers. (Although usually λ is used for Lagrange multipliers, V is used instead because these multipliers turn out to be the nodal voltages). Y will be called the unconstrained power function. The next step is to form the following equations, in accord with the MPL.

$$\partial Y/\partial i_1 = 0 \tag{6}$$

$$\partial Y/\partial i_2 = 0 \tag{7}$$

$$\partial Y/\partial i_3 = 0 \tag{8}$$

$$\partial Y/\partial i_4 = 0 \tag{9}$$

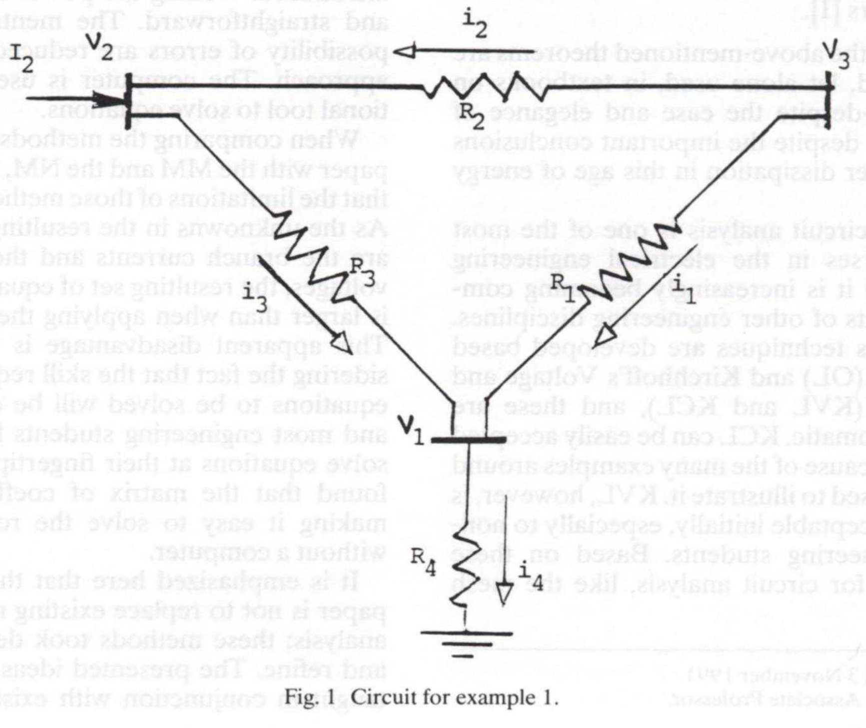

Fig. 1 Circuit for example 1.

These equations, together with the nodal constraint equations, i.e. KCL, give the following linear equations describing the circuit:

$$
\begin{array}{l}
 \\ OL1 \\ OL2 \\ OL3 \\ OL4 \\ KCL1 \\ KCL2 \\ KCL3
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} i_{21} & i_{31} & i_{32} & i_{30} & V_1 & V_2 & V_3 \end{array} \\
\left[\begin{array}{ccccccc}
R_{21} & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & R_{31} & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & R_{32} & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & R_{30} & 0 & 0 & -1 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & 0 & 0 & 0
\end{array}\right]
\end{array}
=
\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I_2 \\ 0 \end{array}\right\}
$$

The first four equations come from eqs (6)–(9) and they are recognized as OL, but this fact is incidental and need not be mentioned here. The last three are KCL.

An alternative to using Lagrange multipliers to minimize F, given by Eq. (1) subject to Eqs (2)–(4), is to use the elimination method. Suppose in this example we are interested in i_2 only, then Eqs (2)–(5) can be written as:

$i_1 = -i_2$

$i_3 = I_2 + i_2$

$i_4 = (-i_2) + (I_2 + i_2)$

F can now be written in terms of i_2 only as:

$$F = R_1 i_2^2 + R_2 i_2^2 + R_3(I_2 + i_2)^2 + R_4 I_2^2 \quad (10)$$

and finding i_2 that minimizes f is straightforward.

COROLLARY 1 *In an AC circuit having no voltage sources, the injected currents will be distributed in the branches such that* $\Sigma i_{pq}^2 Z_{pq}$ *is minimal, subject to KCL, the summation index is over the branches;* i_{pq} *is a phasor and* Z_{pq} *is branch impedance.*

THEOREM 2 *In a circuit containing independent current and voltage sources, the branch currents will be such that* $\Sigma(1/2 i^2 R + Ei)$ *is minimal, where E is a branch emf source with a polarity such that Ei is power absorbed by the source. In case non-linear resistances exist, then* $\Sigma((\int_0^i v(i)\mathrm{d}\, i) + Ei)$ *is minimal, where* $v(i)$ *is the characteristic of the branch resistance.*

The proof of this theorem is given in Appendix II.

APPLICATION

Assume in the previous example that there is an emf source E in branch 21 with a polarity such that i_3E is power absorbed by the source. The power function F to be minimized now is:

$$F = 1/2[R_1 i_1^2 + R_2 i_2^2 + R_3 i_3^2 + R_4 i_4^2] + E i_3 \quad (11)$$

subject to the constraint Eqs (3)–(5). The resulting equations will be the same as those of example 1 with the exception of the right-hand vector, which becomes

$$(-E\ 0\ 0\ 0\ 0\ I_2\ 0)'.$$

If an independent voltage source is connected between a node and the reference node, then the voltage at that node is known and the column corresponding to it can be deleted after modifying the right-hand constant vector accordingly.

GENERAL METHOD FOR NETWORK ANALYSIS

The following steps are suggested for resolving networks:

1. Draw the circuit and assign branch currents such that they enter the positive terminal of the branch emf source, if any. Branches containing current sources are removed and their effect will be presented as nodal current injection, positive and negative, at both ends of the branch, as shown in Figs 2 and 3. Identify the nodes.
2. Write the power function F, according to theorem 2, $F = \Sigma(1/2 Ri^2 + Ei)$.
3. Write KCL at every node, preferably with currents into the nodes considered positive.
4. Minimize F subject to the constraints obtained in step 3.

Dependent sources will be treated as independent during the minimization process. This will result in a set of simultaneous equations with the branch currents as unknowns. The equations will also have $I_{\text{dependent}}$ and $E_{\text{dependent}}$ and must be augmented by the equations describing the dependency of those sources. For example, suppose that in the previous circuit a dependent current source is added at node 3, such that $I_3 = 2V_1 + 3i_{21}$, row 7 now becomes

$$(-3\ 1\ 1\ 1\ -2\ 0\ 0\ 0).$$

The value of I_3 can be calculated if desired by augmenting the previous unmodified matrix by a row

$$(-3\ 0\ 0\ 0\ -2\ 0\ 0\ 0\ 1)$$

and a column $(0\ 0\ 0\ 0\ 0\ 0\ -1\ 1)'$.

To summarize: to find the branch currents and the nodal voltages in a circuit containing both dependent and independent sources, theorem 2 can be applied, treating the dependent sources as independent when minimizing the power function. The resulting equations are augmented by equations describing the dependency of the dependent sources. Thus for each branch an OL-type equation results, for each unknown nodal voltage a KCL is written, and for each dependent source an equation describing its dependency can be written;

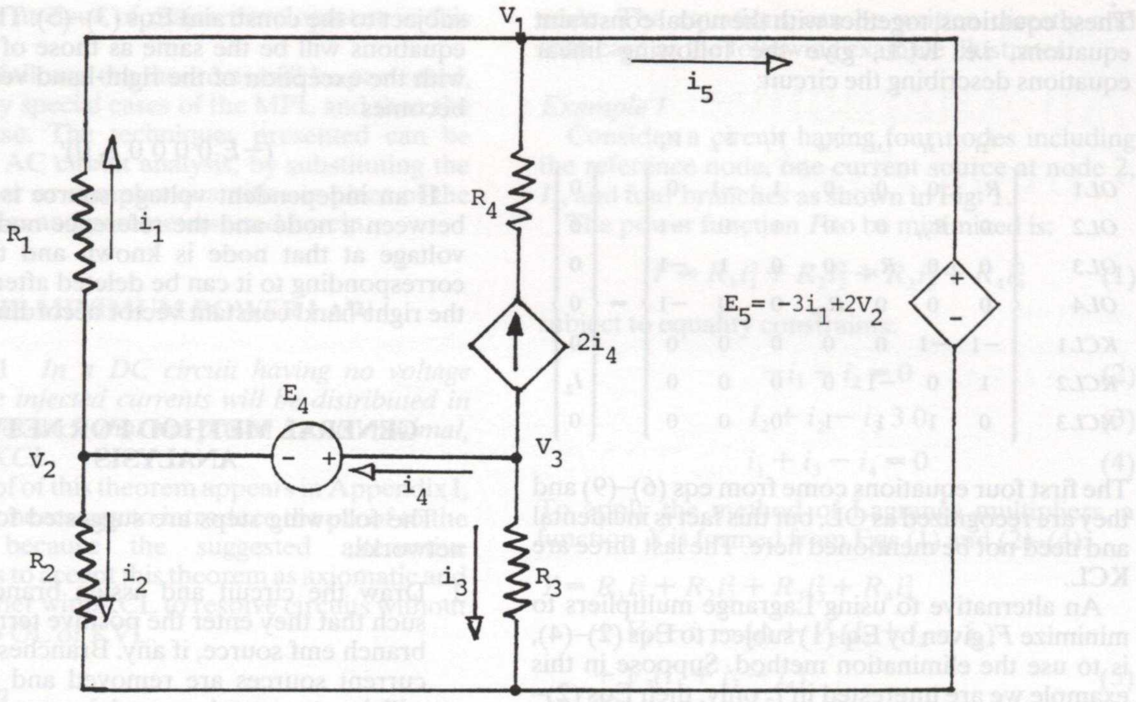

Fig. 2 Circuit for example 2.

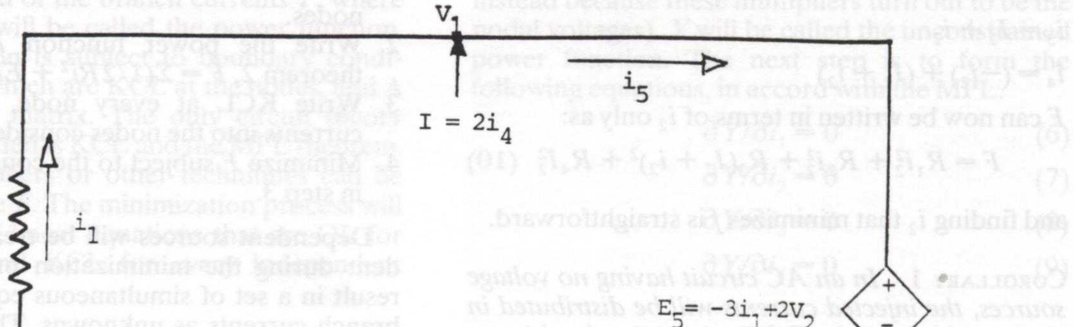

Fig. 3 Redrawn circuit for example 2.

thus, the number of equations is the same as the number of unknowns.

Example 2

Figure 2 shows a circuit to be resolved. It has a voltage source, E_4; a current-controlled current source, $I = 2i_4$; and a current and voltage-controlled voltage source, $E_5 = -3i_1 + 2V_2$. The circuit is redrawn so that the current source appears as an injected current at node 1 and a negative injected current at node 2. The revised circuit is shown in Fig. 3. The power function is:

$$F = 1/2[R_1 i_1^2 + R_2 i_2^2 + R_3 i_3^2] + i_4 E_4 + i_5 E_5 \quad (12)$$

F should be minimized subject to:

$$I + i_1 - i_5 = 0 \quad (13)$$

$$i_4 - i_1 - i_2 = 0 \quad (14)$$

$$-I - i_4 - i_3 = 0 \tag{15}$$

Note that E_4 and I are treated as constants in the minimization process. The unconstrained power function is:

$$\begin{aligned} Y = {} & 1/2[R_1 i_1^2 + R_2 i_2^2 + R_3 i_3^2] + i_4 E_4 + i_5 E_5 \\ & + V_1(I + i_1 - i_5) + V_2(i_4 - i_1 - i_2) \\ & + V_3(-I - i_4 - i_3) \end{aligned} \tag{16}$$

Taking the derivatives with respect to the i's gives five equations:

$$R_1 i_1 + V_1 - V_2 = 0 \tag{17}$$

$$R_2 i_2 - V_2 = 0 \tag{18}$$

$$R_3 i_3 - V_3 = 0 \tag{19}$$

$$E_4 + V_2 - V_3 = 0 \tag{20}$$

$$E_5 - V_1 = 0 \tag{21}$$

The dependent source voltage E_5 may be treated as an unknown and the equation describing its dependency added to the resulting set of equations, or the expression of E_5 in terms of the branch currents and nodal voltages is made use of at this point to get:

$$-3i_1 + 2V_2 - V_1 = 0 \tag{22}$$

Equation (22) replaces (21). The same treatment will be used for the current-controlled current source, I. The above five equations are augmented with the three KCL equations to obtain the eight equations that are needed. These equations written in matrix form are:

$$\begin{array}{l} \\ OL1 \\ OL2 \\ OL3 \\ Br4 \\ Br5 \\ KCL1 \\ KCL2 \\ KCL3 \end{array} \begin{array}{c} \begin{array}{cccccccc} i_1 & i_2 & i_3 & i_4 & i_5 & V_1 & V_2 & V_3 \end{array} \\ \left\{\begin{array}{cccccccc} R_1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & R_2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & R_3 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -3 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 & 0 & 0 & 0 & 0 \end{array}\right\} \end{array} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ E_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right\}$$

CONCLUSION

An alternative approach to resolving circuits containing both dependent and independent sources is presented. The circuit theory tools needed are KCL and the MPL introduced in this paper. A power function, F, is defined and can be easily written for any circuit. Minimizing F subject to KCL yields a set of linear equations, the solution of which provides the branch currents. The method is particularly useful where the nodal method and the mesh method cannot be directly applied.

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REFERENCES

1. J. C. Maxwell, *A Treatise on Electricity Magnetism*, Dover (1954).

APPENDIX I

Proof of theorem 1

PROOF Let the current in the branch from node p to node q be i_{pq}. According to OL $i_{pq} = \{(V_p - V_q)\}/\{R_{pq}\}$. Suppose the current in branch pq is not i_{pq} as given by OL, but $i'_{pq} = i_{pq} + \Delta i_{pq}$ (the currents i' still obey KCL). The power loss in this case is $\Sigma i'^2_{pq} R_{pq} = \Sigma(i_{pq} + \Delta i_{pq})^2 R_{pq}$ and

$$\Sigma i'^2_{pq} R_{pq} = \Sigma R_{pq} i_{pq}(i_{pq} + 2\Delta i_{pq}) + \Sigma(\Delta i_{pq})^2 R_{pq} \tag{23}$$

The first term on the right can be shown to be $\Sigma V_p I_p$, which is the same as $\Sigma i^2_{pq} R_{pq}$, and since the last term on the right is positive, the losses when the currents do not obey OL (i'_p) are therefore greater than when they do.

The above proof is similar to that presented by Maxwell. An alternative proof can be presented using minimization techniques and OL. The theorem also, can be more acceptable as axiomatic than KVL especially for non-electrical engineering students.

APPENDIX II

Proof of theorem 2

Theorem 1 is a special case of theorem 2, and a similar proof can be given. When non-linear resistances are present, the following proof is more general.

PROOF Suppose that the branch currents as determined by KCL and OL are $\mathbf{i_0}$. Define $F(\mathbf{i_0})$ to be

$$\sum \int_0^{i_o} v(i)\mathrm{d}i + Ei;$$

it is required to show that $F(\mathbf{i}') > F(\mathbf{i}_o)$, for any $\mathbf{i}'$ satisfying KCL. Assume that $i'_{pq} = i_{pqo} + \Delta i_{pq}$, then

$$F(i') = \sum \int_0^{i_{pqo}+\Delta i_{pq}} V_{pq}(i_{pq})\mathrm{d}\, i_{pq} \tag{24}$$

where $V_{pq}(i_{pq})$ is the voltage across branch pq.

$$F(i') = \sum \int_0^{i_{pqo}} V_{pq}(i_{pq})\mathrm{d}\, i_{pq} + \sum \int_{i_{pqo}}^{i_{pqo}+\Delta i_{pq}} V_{pq}(i_{pq})\mathrm{d}\, i_{pq} \tag{25}$$

$$F(i') = F(i_o) + \sum \int_{i_{pqo}}^{i_{pqo}+\Delta i_{pq}} V_{pq}(i_{pq})\mathrm{d}\, i_{pq} \tag{26}$$

Since $V_{pq}(i_{pq})$ is an increasing function of i_{pq}

$$\int_{i_{pqo}}^{i_{pqo}+\Delta i_{pq}} V_{pq}(i_{pq})\mathrm{d}\, i_{pq} > V_{pq}(i_{pqo})\Delta i_{pq} \tag{27}$$

regardless of the sign of Δi_{pq}. And since $\Sigma V_{pq}(i_{pqo})\Delta i_{pq} = 0$, because $\mathbf{\Delta i}$ obeys KCL, $\sum \int_{i_{pqo}}^{i_{pqo}+\Delta i_{pq}} V_{pq}(i_{pq})\mathrm{d}\, i_{pq}$ is positive, which proves the theorem.

Applying the MPL to cases where non-linear resistance are present will result in a non-linear set of algebraic equations in $\mathbf{i}$; special techniques must be used to solve these equations.