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Coplanar Projectile Motion Including the Effects of Constant Thrust and Drag*

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The elementary coplanar projectile motion problem is expanded to include a constant thrust and a velocity-dependent drag. Using a method previously developed, this nonlinear problem is shown to possess an exact solution. Several important cases are discussed

INTRODUCTION

IN two previous papers (Gillis and Jones [1] and Jones, Gillis, and Vujanovic [2]), it was demonstrated that the solution to a wide class of projectile motion problems is within the grasp of students in elementary dynamics. These problems are very useful for extended classroom discussion and as projects. They acquaint the student with a nontrivial exercise and bring the classroom experience closer to the real world. At the same time, this class of problems is exactly solvable and allows the student to see the form that solutions to systems of nonlinear differential equations can take. These problems also have the advantage of requiring only a basic knowledge of differential equations. The solutions to these apparently complicated systems can be achieved through a sequence of elementary steps. The result is a parametric representation which applies to a variety of physical problems along with a variety of initial conditions. Information can be easily extracted from the parametric solution because the solutions are algebraic or require, at most, the evaluation of definite integrals involving well behaved integrands.

In this paper, the discussion of these problems is expanded to include the effect of constant thrust as well as drag. The drag functions are proportional to powers of the local velocity. The complete solutions to two problems are given as examples.

THEORY

Consider the coplanar motion of a projectile which is launched into an atmosphere that affords a velocity-dependent resistance $D(v)$. There is negligible loss of mass m, constant gravitational force mg, and constant thrust F. The thrust acts over a prescribed interval of time $0 \leqslant t \leqslant \bar{t}$. The thrust and the drag are in the direction of the local tangent to the trajectory. Under these conditions, the equations of motion are

$$m\ddot{x} = (F - D)\dot{x}/v \tag{1}$$

and

$$m\ddot{y} = -mg + (F - D)\dot{y}/v \tag{2}$$

where dots over symbols denote differentiation with respect to time.

This system is generally nonlinear and must be integrated subject to the initial conditions

$$x(0) = 0, y(0) = h \tag{3}$$

and

$$\dot{x}(0) = v_0\cos\theta, \dot{y}(0) = v_0\sin\theta \tag{4}$$

where h is the launch elevation (see Fig. 1), v_0 is the launch speed, and θ is the launch angle.

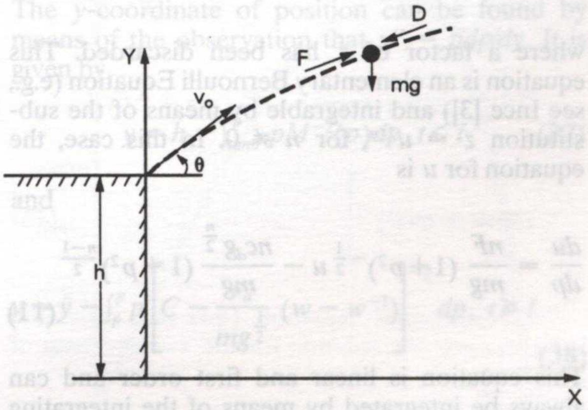

Fig. 1. Coplanar motion of a projectile in an atmosphere with drag $D = D(v)$. The propulsive force F is constant and there is negligible loss of mass m.

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As pointed out in Jones, *et al*. [2], an exact solution in terms of elementary functions can be found for the case when $F = 0$ (unpowered flight). The technique used was an analysis of the motion along the trajectory $y = y(x)$. The same approach will be adopted in this paper.

Suppose that the drag force $D = D(v)$ can be expressed as a power function

$$D = D(v) = c_d v^n. \tag{5}$$

where n is a dimensionless exponent and c_d is a constant drag coefficient with the dimension of force per unit $(\text{velocity})^{-n}$.

Consider the motion of the projectile (1), (2) along the trajectory $y = y(x)$. Differentiating the trajectory equation, we get

$$\dot{y} = y'\dot{x} \tag{6}$$

and

$$\ddot{y} = y'\ddot{x} + y''\dot{x}^2 \tag{7}$$

where $y' \equiv dy/dx$ and $y'' \equiv d^2y/dx^2$. By combining (1), (2), (6) and (7), we get a fundamental relationship for the curvature of the trajectory.

$$y'' = -g/\dot{x}^2. \tag{8}$$

This equation can be differentiated and combined with (1) and (2) to get the equation of motion of the projectile along the trajectory.

$$y''' = \frac{2y''^2}{mg}\left[F - c_d(-g/y'')^{\frac{n}{2}}(1+y'^2)^{\frac{n}{2}}\right](1+y'^2)^{-\frac{1}{2}} \tag{9}$$

The integration of this equation can be accomplished by means of the substitution $y' = p$, $y'' = -z$, and $y''' = zdz/dp$. With this substitution, (9) becomes

$$\frac{dz}{dp} = \frac{2F}{mg}(1+p^2)^{-\frac{1}{2}}z - \frac{2c_d g^{\frac{n}{2}}}{mg}(1+p^2)^{\frac{n-1}{2}}z^{\frac{2-n}{2}} \tag{10}$$

where a factor of z has been discarded. This equation is an elementary Bernoulli Equation (e.g., see Ince [3]) and integrable by means of the substitution $z = u^{2/n}$, for $n \neq 0$. In this case, the equation for u is

$$\frac{du}{dp} = \frac{nF}{mg}(1+p^2)^{-\frac{1}{2}}u - \frac{nc_d g^{\frac{n}{2}}}{mg}(1+p^2)^{\frac{n-1}{2}} \tag{11}$$

This equation is linear and first order and can always be integrated by means of the integrating factor

$$(p + (1+p^2)^{\frac{1}{2}})^{-\frac{nF}{mg}} \tag{12}$$

Multiplication of (11) by (12) results in an exact differential equation that can be immediately integrated to give

$$u = z^{\frac{n}{2}} = Cw^{\frac{nF}{mg}} - \frac{c_d g^{\frac{n}{2}}}{mg} w^{\frac{nF}{mg}} \times \int (1+p^2)^{\frac{n-2}{2}} w^{-\frac{nF}{mg}} dp \tag{13}$$

where C is an arbitrary constant of integration and

$$w = p + (1+p^2)^{\frac{1}{2}} \tag{14}$$

In many instances, the integral appearing in (13) can be evaluated in terms of elementary functions. Several of these important cases will be discussed in succeeding sections.

Powered flight in a vacuum

If drag is neglected, then $c_d = 0$ and (13) reduces to

$$z = -y'' = Cw^{\frac{2F}{mg}} \tag{15}$$

where $w = w(p)$ is defined in equation (14). The constant C can be evaluated from the initial conditions

$$y(0) = h \tag{16}$$

$$y'(0) = tan\theta \tag{17}$$

$$y''(0) = -gv_0^{-2}sec^2\theta \tag{18}$$

and

$$C = gv_0^{-2}sec^2\theta(tan\theta + sec\theta)^{-\frac{2F}{mg}} \tag{19}$$

Notice that the projectile must be launched with nonzero initial velocity, even though the flight is powered.

The differential equation for y in (15) has separable variables and can be integrated once by observing that $y' = p$ and $y'' = dp/dx$. The result of the integration is the x-coordinate of position for the projectile

$$x = -C^{-1}\int_{tan\theta}^{p} w^{-\frac{2F}{mg}} dp, \quad t < \bar{t} \tag{20}$$

and

$$x = \bar{x} - C^{-1}(p - \bar{p}), \quad t \geqslant \bar{t} \tag{21}$$

where $\bar{p}$ is the slope of the tangent to the trajectory at the point when $t = \bar{t}$ is reached. $\bar{t}$ is the time at which powered flight ceases or $F = 0$. $\bar{x}$ is the x-coodinate of position when powered flight ceases. This is given by

$$\bar{x} = -C^{-1}\int_{tan\theta}^{\bar{p}} w^{-\frac{2F}{mg}} dp. \tag{22}$$

The y-coordinate of position can be found by means of the observation $y'' = pdp/dy$.

$$y = h - C^{-1} \int_{tan\theta}^{p} pw^{-\frac{2F}{mg}} dp, \; t < \bar{t} \qquad (23)$$

and

$$y = \bar{y} - \frac{1}{2} C^{-1}(p^2 - \bar{p}^2), \; t \geqslant \bar{t} \qquad (24)$$

where $\bar{y}$ is the y-coordinate of position when powered flight ceases. This is given by

$$\bar{y} = h - C^{-1} \int_{tan\theta}^{\bar{p}} pw^{-\frac{2F}{mg}} dp. \qquad (25)$$

Equations (20) and (23) represent the parametric solution to the problem in terms of the parameter p when $t < \bar{t}$. Equations (21) and (24) represent the parametric solution to the problem in terms of the parameter p when $t \geqslant \bar{t}$. Notice that (21) and (24) can be combined to give the classic parabolic trajectory by algebraically eliminating p between (21) and (24). The integrals that appear in all of these equations can be evaluated in terms of elementary functions. The details of the integration are contained in the Appendix and a detailed example later in the paper.

To complete the solution, we need to find $\bar{p}$ given the time for powered flight $\bar{t}$. An integral for current time can be developed directly from (8) in terms of the tangent parameter p. Extracting the root in (8) and separating leads to

$$dt = (-y''/g)^{\frac{1}{2}} dx. \qquad (26)$$

By combining this equation with (15) and (20), we get

$$dt = -(gC)^{-\frac{1}{2}} w^{-\frac{F}{mg}} dp. \qquad (27)$$

This equation can be integrated directly and the result is

$$t = -(gC)^{-\frac{1}{2}} \int_{tan\theta}^{p} w^{-\frac{F}{mg}} dp, \; t < \bar{t} \qquad (28)$$

and

$$t = \bar{t} - (gC)^{-\frac{1}{2}} (p - \bar{p}), \; t \geqslant \bar{t} \qquad (29)$$

where

$$\bar{t} = -(gC)^{-\frac{1}{2}} \int_{tan\theta}^{\bar{p}} w^{-\frac{F}{mg}} dp. \qquad (30)$$

The last equation can be used to find $\bar{p}$ and this value of the tangent parameter can then be used to find $\bar{x}$ and $\bar{y}$ in (22) and (25). The solution of the powered flight problem in a vacuum is now complete.

Powered flight with viscous drag

In this case, $n = 1$ and the drag function (5) becomes

$$D = D(v) = c_d v. \qquad (31)$$

Equation (13) reduces to

$$z^{\frac{1}{2}} = (-y'')^{\frac{1}{2}} = Cw^{\frac{F}{mg}} - \frac{c_d}{mg^{\frac{1}{2}}} w^{\frac{F}{mg}} \int (1 + p^2)^{-\frac{1}{2}}$$

$$w^{-\frac{F}{mg}} dp = Cw^{\frac{F}{mg}} - \frac{c_d}{mg^{\frac{1}{2}}} \left(\frac{mg}{mg - F} w - \frac{mg}{mg + F} w^{-1} \right) = M(p). \qquad (32)$$

Where w is defined by (14), and $F \neq mg$. The constant C in this equation can be evaluated from (16)–(18) and the result is

$$C = (tan\theta + sec\theta)^{-\frac{F}{mg}} \left\{ g^{\frac{1}{2}} v_0^{-1} sec\theta + \frac{c_d}{2mg^{\frac{1}{2}}} \left[\frac{mg}{mg - F} (tan\theta + sec\theta) - \frac{mg}{mg + F} (tan\theta + sec\theta)^{-1} \right] \right\} \qquad (33)$$

Equation (32) has separable variables. Separation and integration lead to an expression for the x-coordinate of position for the projectile.

$$x = -\int_{tan\theta}^{p} M^{-2}(p) dp, \; t < \bar{t} \qquad (34)$$

and

$$x = \bar{x} - \int_{\bar{p}}^{p} \left[C - \frac{c_d}{mg^{\frac{1}{2}}} (w - w^{-1}) \right]^{-2} dp, \; t \geqslant \bar{t} \qquad (35)$$

where

$$\bar{x} = -\int_{tan\theta}^{\bar{p}} M^{-2}(p) dp. \qquad (36)$$

The y-coordinate of position can be found by means of the observation that $y'' = pdp/dy$. It is given by

$$y = h - \int_{tan\theta}^{p} pM^{-2}(p) dp, \; t < \bar{t} \qquad (37)$$

and

$$y = \bar{y} - \int_{\bar{p}}^{p} p \left[C - \frac{c_d}{mg^{\frac{1}{2}}} (w - w^{-1}) \right]^{-2} dp, \; t \geqslant \bar{t} \qquad (38)$$

where

$$\bar{y} = -\int_{tan\theta}^{\bar{p}} pM^{-2}(p) dp. \qquad (39)$$

In all of the above equations, $F \neq mg$. The case for $F = mg$ can be handled separately and that solution will not be treated here.

The time of flight integral is given by

$$t = \frac{-1}{g^{\frac{1}{2}}} \int_{tan\theta}^{p} M^{-1}(p)dp, t < \bar{t} \tag{40}$$

and

$$t = \bar{t} - \frac{1}{g^{\frac{1}{2}}} \int_{\bar{p}}^{p} \left[C - \frac{c_d}{mg^{\frac{1}{2}}} (w - w^{-1}) \right]^{-1} dp, t \geqslant \bar{t} \tag{41}$$

where

$$\bar{t} = \frac{-1}{g^{\frac{1}{2}}} \int_{tan\theta}^{\bar{p}} M^{-1}(p)dp \tag{42}$$

can be used to find $\bar{p}$. This completes the solution of the powered flight problem with viscous drag.

Powered flight with velocity–squared drag

In this case, $n = 2$ and the drag function (5) becomes

$$D = D(v) = c_d v^2. \tag{43}$$

Equation (13) reduces to

$$z = -y'' = Cw^{\frac{2F}{mg}} - \frac{gc_d}{2}(mg - 2F)^{-1} w + \frac{gc_d}{2}(mg + 2F)^{-1} w^{-1} = G(p) \tag{44}$$

where w is defined in (14) and $2F \neq mg$. The case for $2F = mg$ can be handled separately and will not be presented here. The constant C appearing in (44) can be evaluated with (16)–(18).

$$C = (tan\theta + sec\theta)^{-\frac{2F}{mg}} \left[gv_0^{-2} sec^2\theta + \frac{c_d g}{2}(mg - 2F)^{-1}(tan\theta + sec\theta) - \frac{c_d g}{2}(mg + 2F)^{-1}(tan\theta + sec\theta)^{-1} \right] \tag{45}$$

The differential equation (44) has separable variables and the solution for the x-coordinate of position is

$$x = -\int_{tan\theta}^{p} [G(p)]^{-1} dp, t < \bar{t} \tag{46}$$

and

$$x = \bar{x} - \int_{\bar{p}}^{p} \left[C - \frac{c_d}{2m} w + \frac{c_d}{2m} w^{-1} \right]^{-1} dp, t \geqslant \bar{t}. \tag{47}$$

For the y-coordinate of position, we have

$$y = h - \int_{tan\theta}^{p} p[G(p)]^{-1} dp, t < \bar{t} \tag{48}$$

and

$$y = \bar{y} - \int_{\bar{p}}^{p} p \left[C - \frac{c_d}{2m} w + \frac{c_d}{2m} w^{-1} \right]^{-1} dp, t \geqslant \bar{t}. \tag{49}$$

The time of flight can be found for the velocity-squared drag case by using (26). The result is

$$t = -g^{-\frac{1}{2}} \int_{tan\theta}^{p} [G(p)]^{-\frac{1}{2}} dp, t < \bar{t} \tag{50}$$

and

$$t = \bar{t} - g^{-\frac{1}{2}} \int_{\bar{p}}^{p} \left[C - \frac{c_d}{2m} w + \frac{c_d}{2m} w^{-1} \right]^{\frac{1}{2}} dp, t \geqslant \bar{t}. \tag{51}$$

The value of $\bar{p}$ can be found from

$$\bar{t} = -g^{-\frac{1}{2}} \int_{tan\theta}^{\bar{p}} [G(p)]^{-\frac{1}{2}} dp. \tag{52}$$

This completes the solution of the velocity-squared drag problem.

Example

For powered flight in a vacuum, using the Appendix, we can show that all of the integrals appearing in that section can be evaluated exactly. When F does not equal $mg/2$ or mg, the exact solution for $t < \bar{t}$ is

$$x = \frac{C^{-1}}{2} \left[\frac{mg}{mg - 2F} \left(w^{1 - \frac{2F}{mg}} - w_0^{1 - \frac{2F}{mg}} \right) - \frac{mg}{mg + 2F} \left(w^{-(1 + \frac{2F}{mg})} - w_0^{-(1 + \frac{2F}{mg})} \right) \right] \tag{53}$$

$$y = h - \frac{C^{-1}}{8} \left[\frac{mg}{mg - F} \left(w^{2 - \frac{2F}{mg}} - w_0^{2 - \frac{2F}{mg}} \right) + \frac{mg}{mg + F} \left(w^{-(2 + \frac{2F}{mg})} - w_0^{-(2 + \frac{2F}{mg})} \right) \right] \tag{54}$$

where w is defined in terms of p in equation (14) and w_o is the value of w at $p = \tan\theta$. The cases for $F = mg/2$ and $F = mg$ can be treated separately and lead to analytical solutions. These cases will not be included in this discussion.

When t reaches $\bar{t}$, $p = \bar{p}$, and the transitional values of x and y, $\bar{x}$ and $\bar{y}$, are given by

$$\bar{x} = \frac{C^{-1}}{2}\left[\frac{mg}{mg-2F}\left(\bar{w}^{1-\frac{2F}{mg}} - w_0^{1-\frac{2F}{mg}}\right) - \frac{mg}{mg+2F}\left(\bar{w}^{-(1+\frac{2F}{mg})} - w_0^{-(1+\frac{2F}{mg})}\right)\right] \quad (55)$$

$$\bar{y} = h - \frac{C^{-1}}{8}\left[\frac{mg}{mg-F}\left(\bar{w}^{2-\frac{2F}{mg}} - w_0^{2-\frac{2F}{mg}}\right) + \frac{mg}{mg+F}\left(\bar{w}^{-(2+\frac{2F}{mg})} - w_0^{-(2+\frac{2F}{mg})}\right)\right] \quad (56)$$

where $\bar{w}$ is the value of w at $p = \bar{p}$.

This completes the solution for the zero-drag case. It is completely analytical. A typical calculation using (53–56) and the post-propulsion solution (21) and (24) is shown in Fig. 2.

For this example, $\theta = 45°$, $v_0 = 20$ m/s, $h = 0$, F $= 1000$N, $\bar{t} = 2$ sec., and the mass of the projectile is 5 kg. It is somewhat difficult to discern, but the trajectory is virtually linear during the propulsive phase. After the propulsive phase has been completed, the trajectory follows the normal parabolic path dictated by (21) and (24). However, the range and angle of impact have been altered by the initial constant thrust.

Example

Consider the powered flight problem with viscous drag discussed by means of equations of (31)–(42). In this case, the integrals cannot be evaluated by means of the Appendix. However, the integrands of all of these integrals are well behaved and numerical evaluation of them can even be achieved on a pocket calculator. A typical calculation is given by Fig. 3. For this example, $c_d = 0.24$ Newton-sec/meter and the thrust $F = 1000$ Newtons. The burn time $\bar{t} = 3.0$ sec. and the mass was launched at an angle $\theta = 45°$ with initial velocity $v_0 = 20$ m/s from $h = 0$. The trajectory shows the typical unsymmetrical profile associated with nonzero drag. However, because the thrust dominates the weight and drag on the protectile during the initial phase of motion, the trajectory is virtually linear.

Several cases for viscous drag are presented in Fig. 4. For some launch conditions ($\theta = 45°$ $v_0 = 20$ m/s thrust $F = 1000$ Newtons and burn time $\bar{t} = 3.0$ sec.) the trajectories of a 5 kg. projectile are compared to the same case for nonzero drag. The drag coefficients are $c_d = 0.10, 0.20, 0.30$ Newton-seconds/meter. Notice that all of the trajectories virtually coincide during the nonzero thrust phase of the motion and the trajectories are nearly linear.

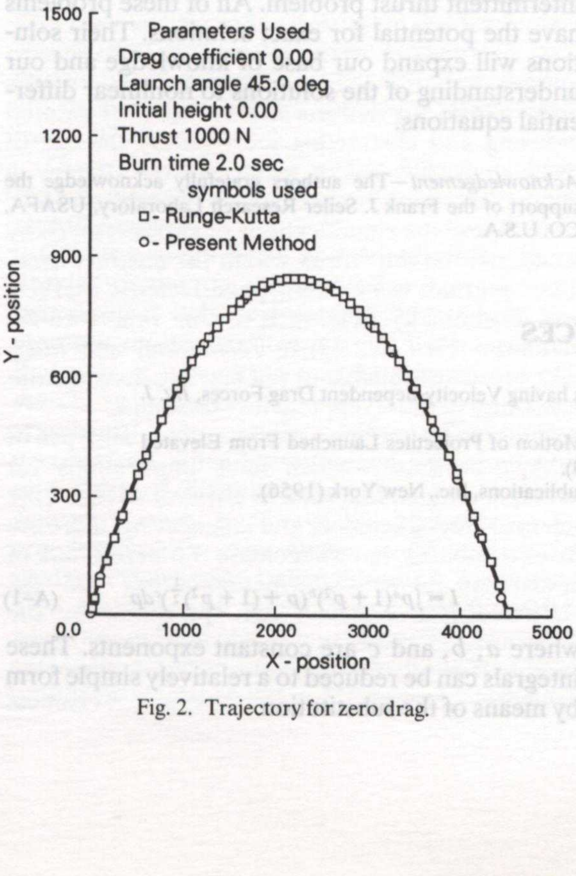

Fig. 2. Trajectory for zero drag.

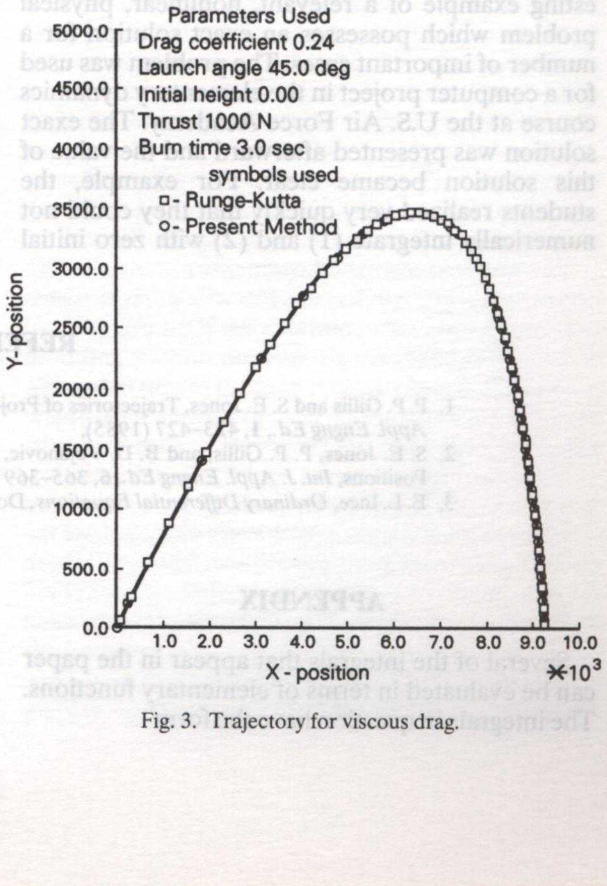

Fig. 3. Trajectory for viscous drag.

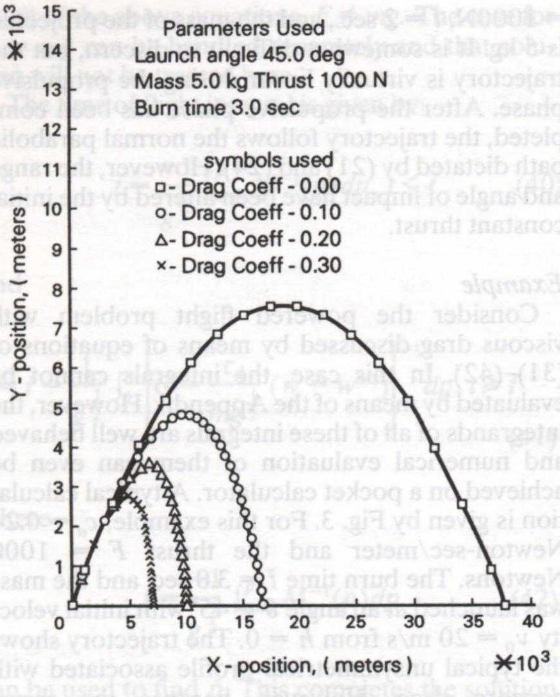

Fig. 4. Trajectories for various values of c_d.

Even for a very large drag coefficient $c_d = 0.3$, the thrust still dominates the drag and the weight. This will be the case until the speed is sufficiently high to produce a drag large enough to reduce the net force along the trajectory and allow the weight to add significantly to the vector sum.

CONCLUSION

In this paper, we have presented another interesting example of a relevant, nonlinear, physical problem which possesses an exact solution for a number of important cases. The problem was used for a computer project in the elementary dynamics course at the U.S. Air Force Academy. The exact solution was presented afterward and the value of this solution became clear. For example, the students realized very quickly that they could not numerically integrate (1) and (2) with zero initial velocity, even though it seemed that the initial thrust would be enough to launch the mass. However, it became perfectly clear that v_0 must be different from zero when the initial conditions are used to evaluate the constant of integration C in (19), (33), and (45).

This exercise further revealed to the students the actual mathematical structure of the solution. As noted earlier [1,2], many of these complex problems possess exact solutions when the equations of motion are expressed in terms of the proper variables.

In this case, the natural variable p, the local slope of the tangent vector, is the key to reducing the system to an integrable form. The transformation of (9) to the linear first order equation (11) taught the students that some complicated nonlinear systems are often thinly veiled examples of elementary problems. The solution of these elementary problems is within their grasp with only the first course in differential equations behind them.

Even when the exact solution required numerical integration, it offered a positive alternative to standard numerical integration. The integrals are easy to evaluate because they all have well behaved integrands. The solution can be computed to any degree of accuracy at any point on the trajectory. Upper and lower bounds on the integrals can be achieved with something as elementary as Simpson's Rule, which the students learned in their first calculus course. It is easy to apply because the slope is generally positive until maximum height is reached, at which point $p = 0$. From that point on, the slopes are all negative.

Further efforts in this area will concentrate on improving the formulation by including the effects of mass loss due to propellant burn, optimizing the range with burn time and thrust, and examining the intermittent thrust problem. All of these problems have the potential for exact solutions. Their solutions will expand our base of knowledge and our understanding of the solutions to nonlinear differential equations.

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APPENDIX

Several of the integrals that appear in the paper can be evaluated in terms of elementary functions. The integrals in question have the form

$$I = \int p^a(1+p^2)^b(p + (1+p^2)^{\frac{1}{2}})^c dp \qquad \text{(A-1)}$$

where a, b, and c are constant exponents. These integrals can be reduced to a relatively simple form by means of the substitution

$$w = p + (1 + p^2)^{\frac{1}{2}} \quad \text{(A–2)}$$

Differentiating this equation, we get

$$dw = (1 + p^2)^{-\frac{1}{2}} w\,dp. \quad \text{(A–3)}$$

Now,

$$p = \frac{1}{2}(w - w^{-1}) \quad \text{(A–4)}$$

and

$$1 + p^2 = \frac{1}{4}(w + w^{-1})^2. \quad \text{(A–5)}$$

Using these relations in (A–1), we find that I now has the form

$$I = 2^{-(1+a+2b)} \int (w - w^{-1})^a (w + w^{-1})^{2b+1} w^{c-1}\,dw. \quad \text{(A–6)}$$

There are numerous integrable cases. For example, whenever a and $2b + 1$ are positive integers I reduces to the evaluation of several integrals involving only power functions.