Int. J. Appl. Engng Ed. Vol. 7, No. 4, pp. 289–293, 1991
Printed in Great Britain.

0742–0269/91 $3.00+0.00

A Finite Element Program for Teaching Transient Axisymmetric Field Problems*

D. S. JAYAS†
G. E. LALIBERTE
K. ALAGUSUNDARAM
Department of Agricultural Engineering, University of Manitoba, Winnipeg, Canada R3T 2N2

A finite element program was written to solve transient axisymmetric heat transfer problems. The program is capable of handling prescribed boundary temperatures, surface convections, external fluxes across the boundary and internal heat generation within the body. The program uses a finite element procedure for temperature distribution at any time step and Euler's backward difference technique in time for solution of the transient problem. The final set of equations were solved by the Gauss elimination method. The program is added as 10th option to FEMPAC (a Finite Element Package of learner programs developed by Dr. R. J. Gustafson for teaching fundamentals of finite elements). A copy of FEMPAC with associated documentation can be obtained from the authors or Dr R. J. Gustafson at the Ohio State University.

INTRODUCTION

A COURSE on the solution of partial differential equations usually forms a part of the academic program of students in many speciality areas of Agricultural Engineering. Steady-state and transient problems in heat transfer, gaseous diffusion and mass transfer are governed by partial differential equations. Depending on the complexity of the problems, they are also classified as one, two or three dimensional. One group of three dimensional problems are axisymmetric field problems. The symmetry about the axis makes it possible to describe these problems using only two views (front and top) as compared to the other three dimensional problems which require three views (front, top and side). Cooling of the grain stored in a cylindrical grain bin by inserting a pipe in the centre of the grain mass and blowing cool air through the pipe is an example of an axisymmetric heat transfer problem. Some other physical problems such as flow of water to an auger hole and calculation of insulation requirements for steam-carrying pipes are also axisymmetric field problems. Many of the agricultural products, such as tomatoes, apples, oranges, carrots and pears may not be exactly axisymmetric but can be approximated as axisymmetric for analysis purposes. Cooling and heating of all these products can be described by axisymmetric analysis.

The objectives of this paper were to develop a finite element program capable of solving axisymmetric field problems, to compare results from the program with other analytical solutions of simple problems and to illustrate the use of the program in teaching a course dealing with the solution of axisymmetric field problems.

* Paper accepted 12 November 1990.
† Associate Professor.

MATHEMATICAL STATEMENT OF THE PROBLEM

Any transient field problem (as an example, heat transfer problem) in cylindrical coordinates is governed by the following partial differential equation [1]:

$$K_{rr}\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}K_{rr}\frac{\partial\phi}{\partial r}+K_{\theta\theta}\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+K_{zz}\frac{\partial^2\phi}{\partial z^2}+Q=\rho c\frac{\partial\phi}{\partial t} \quad (1)$$

with the boundary conditions

$$\phi=\phi_1 \text{ at } S_1 \quad (2)$$

and

$$K_{rr}\frac{\partial\phi}{\partial r_l}+K_{\theta\theta}\frac{\partial\phi}{\partial\theta_l}+K_{zz}\frac{\partial\phi}{\partial z_l}+q+h(\phi-\phi_\infty)=0 \text{ at } S_2 \quad (3)$$

where: r, θ and z are cylindrical coordinate directions,
$K_{rr}, K_{\theta\theta}, K_{zz}$ are thermal conductivities of the medium in r, θ, and z directions, respectively, W/m · K (Btu/h · ft · °F),
ϕ is temperature at any time and position, K (°F),
ρ is the mass density of the material, kg/m^3 (lbm/ft^3),

c	is the specific heat of the material, J/kg · K (Btu/lbm · °F)
Q	is the internal heat generation per unit volume within the body, W/m³ (Btu/h · ft³)
t	is the time, s (h)
S_1 and S_2	are boundary segments for the body and both combined together make the total boundary of the system,
l_r, l_θ, l_z	are the direction cosines,
q	is the externally applied flux to the body, W/m² (Btu/h · ft²),
h	is the convective heat transfer coefficient, W/m² · K (Btu/h · ft²°F), and
ϕ_∞	is the fluid temperature at convection side, K (°F).

Both boundary conditions may occur simultaneously in a problem but may not occur on the same boundary segments. The convection and externally applied flux may occur across the same boundary.

For an axisymmetric heat transfer problem, the temperature at any time is independent of angle θ, therefore, equation (1) with its associated boundary conditions is reduced to:

$$K_{rr}\frac{\partial^2\phi}{\partial r^2}+\frac{K_{rr}}{r}\frac{\partial\phi}{\partial r}+K_{zz}\frac{\partial^2\phi}{\partial z^2}+Q=\rho c\frac{\partial\phi}{\partial t} \quad (4)$$

with the boundary conditions

$$\phi=\phi_1 \text{ at } S_1 \quad (5)$$

and

$$K_{rr}\frac{\partial\phi}{\partial r_{l_r}}+K_{zz}\frac{\partial\phi}{\partial z_{l_z}}+q+h(\phi-\phi_\infty)=0 \text{ at } S_2. \quad (6)$$

To solve equations (4) to (6) using the finite element method, two approaches (variational and weighted residuals) can be used. The details on these approaches are given in finite element books [2,3], respectively. To use the variational approach, a functional must either be available or be derived whereas the weighted residual approach can be used starting with the partial differential equation. Since the functional for the problems described by equations (4) to (6) was available, we used the variational approach.

The functional for an axisymmetric heat transfer problem described by equations (4), (5) and (6) is given by [2]:

$$F=\int_V\frac{1}{2}\left[rK_{rr}\left(\frac{\partial\phi}{\partial r}\right)^2+rK_{zz}\left(\frac{\partial\phi}{\partial z}\right)^2-2r\left(Q-\rho c\frac{\partial\phi}{\partial t}\right)\phi\right]dV+\int_{S_2}q\phi\, ds \quad (7)$$

$$+\int_{S_3}\frac{h}{2}[(\phi-\phi_\infty)^2]ds.$$

The surface integrals are over surfaces S_2 and S_3. The surfaces have been shown as S_2 and S_3 instead of S_2 alone as in equation (6) because of the fact that convection and external applied flux may occur at the same boundary segment or may occur along different boundary segments. In other words, part of S_2 and S_3 may also be common. The evaluation of integrals would be performed over the surfaces along which the convection is prevailing or the flux is being transferred. Therefore, from here onward, the subscripts on S_2 and S_3 will be dropped and S would be used in general terms.

When the functional F is minimized with respect to nodal temperatures $\{\phi\}$ the following set of equations results (Segerlind, 1976):

$$[C]\frac{\partial\{\phi\}}{\partial t}+[K]\{\phi\}+\{F\}=0 \quad (8)$$

where:

$$[C]=\int_v r\rho c[N]^T[N]dV \quad (9)$$

$$[K]=\int_v[B]^T[D][B]dV+\int_s h[N]^T[N]dS \quad (10)$$

$$[F]=-\int_v rQ[N]^T dV+\int_s q[N]^T dS-\int_s h\phi_\infty[N]^T dS \quad (11)$$

[D] = material property matrix,
[N] = interpolation function matrix,
[B] = matrix relating nodal potentials to the gradients,
[K] = conduction matrix,
[C] = capacitance matrix, and
[F] = force vector.

In equation (8), the derivative term was approximated by Euler's backward difference technique. This resulted in a set of equations (12).

$$[K]\{\phi''\}+\frac{[C]\{\phi''\}}{\Delta t}-\frac{[C]\{\phi'\}}{\Delta t}=\{F\} \quad (12)$$

where the superscripts ″ and ′ indicate values of temperature at time t and $t+\Delta t$, respectively.

THE COMPUTER PROGRAM

To solve the set of equations (12), a finite element program was written in FORTRAN. The program can be used to analyze axisymmetric transient heat transfer problems with specified boundary conditions. The program is capable of handling prescribed boundary temperatures, surface convections, external fluxes across the boundary and internal heat generation within the body. The user must specify the initial temperature for each node. The time step, number of time steps and the frequency of printing of nodal temperatures can be controlled by the user. The program uses a finite element procedure for temperature distribution at any time step and Euler's backward difference technique in time for solution of the transient problem. The final set of equations were solved by the Gauss elimination method.

The program uses linear or quadratic triangular elements. The automatic grid generation program of Segerlind (1988, personal communication) can be used with this program.

The program is written in a way so that it can be added as an option in the FEMPAC (Finite Element Method Package) developed by R. J. Gustafson [4]. Detailed input-output description for the program along with listing of the program can be obtained from the authors.

AXISYMMETRIC TRANSIENT HEAT TRANSFER EXAMPLE

Sample Problem Solved Using Heisler's Charts

A sample problem given in Pitts and Sissom [5] was used to test the results of the program. The problem states 'A solid mild steel, 2 in diameter by 2.5 in long cylinder, initially at 1200°F is quenched during heat treatment in a fluid at 200°F. The surface heat transfer coefficient is 150 Btu/h · ft² · °F. Determine the centerline temperature at the midpoint of the length after 2.7 min of immersion.'

This problem is first solved using Heisler's charts following the method given in Pitts and Sissom [5]. The axial conduction is treated by assuming the cylinder to be a slab of thickness 2L (the length of the cylinder), but infinite in the y and z directions. The radial conduction is treated by assuming the

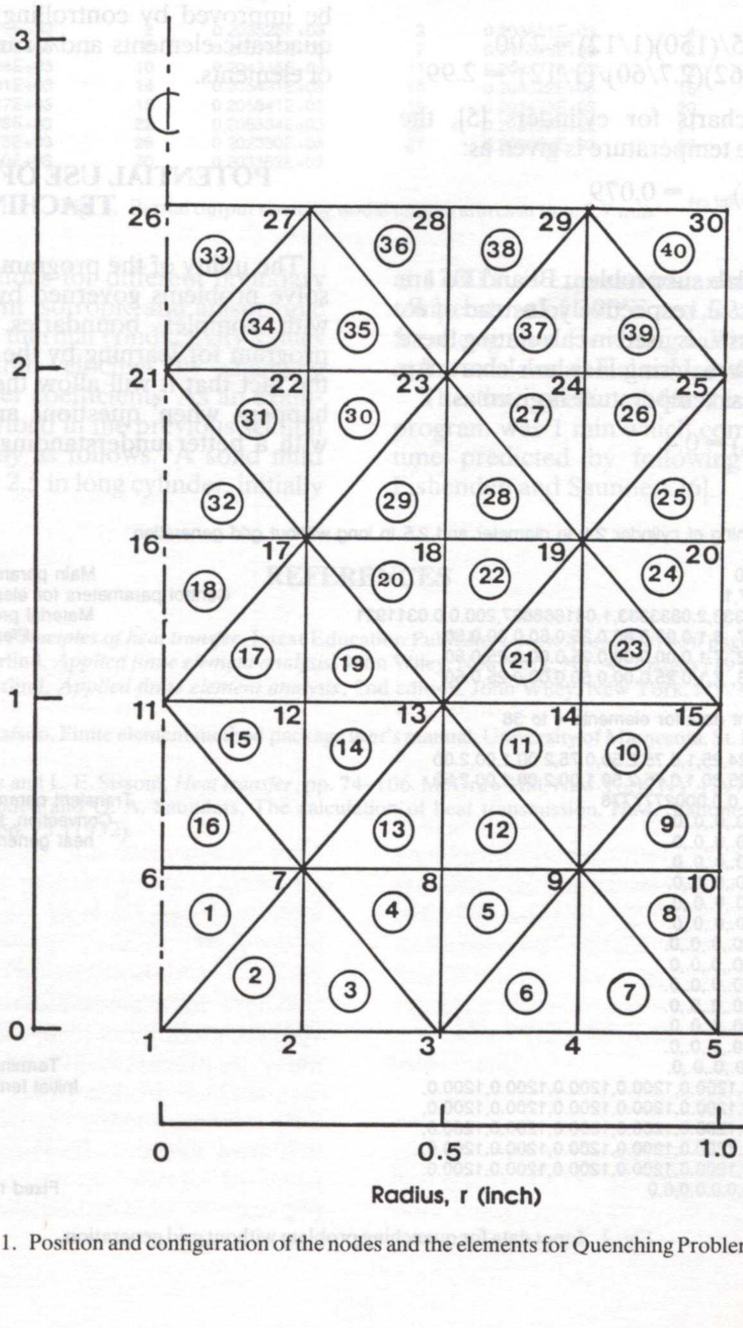

Fig. 1. Position and configuration of the nodes and the elements for Quenching Problem.

rod to be of finite radius R and infinite in length. From the solutions of these slab and cylinder subproblems, the final dimensionless ratio of temperatures is obtained by multiplying these ratios for slab and cylinder.

The material properties at the centre point temperature of 572°F, as taken from Pitts and Sissom [5], are given below.

$$K_{572} = 25\ \text{Btu/h} \cdot \text{ft} \cdot {}^\circ\text{F}$$
$$\alpha_{572} = 0.462\ \text{ft}^2/\text{h}$$

where K is the thermal conductivity of mild steel assumed same in both directions and α is the thermal diffusivity of mild steel. K and α need not to be constant and could vary from element to element.

Cylindrical subproblem

First Biot (Bi) and Fourier (Fo) numbers are calculated from material properties and the radius (R) of the cylinder.

$$1/\text{Bi} = \text{K}/hR = 25/(150)(1/12) = 2.00$$
$$\text{Fo} = \alpha t/R^2 = (0.462)(2.7/60)/(1/12)^2 = 2.99$$

Using Heisler's charts for cylinders [5], the dimensionless centre temperature is given as:

$$(\phi_c - \phi_\infty)/(\phi_i - \phi_\infty)_{\text{inf cyl}} = 0.079$$

Slab subproblem

Similarly for the slab subproblem Bi and Fo are found 1.60 and/or 1.92, respectively. Instead of R, the half slab thickness L is used in calculating these dimensionless numbers. Using Heisler's charts for slab the dimensionless temperature is given as:

$$(\phi_c - \phi_\infty)/(\phi_i - \phi_\infty) = 0.4$$

The dimensionless temperature at the radial and axial centre of the cylinder is 0.032 and, therefore,

$$\phi_c = 200 + 0.032\ (1200\text{–}200) = 232^\circ\text{F}$$

Sample Problem Solution Using the Developed Program

The problem stated above was solved using the axisymmetric transient heat transfer (AXTRHT) program. The region was divided into 40 elements as shown in Fig. 1. The input data for the discretization of Fig. 1 was prepared (Fig. 2) and the problem was solved by executing the program.

Only the partial output is shown in Fig. 3. In addition to the output shown in Fig. 3, the program also prints out most of the input for checking purposes.

The results obtained by finite element formulation are within 11% of the results obtained by Heisler's charts. The solutions from the latter are also approximate. The accuracy of the results could be improved by controlling the time step, using quadratic elements and/or increasing the number of elements.

POTENTIAL USE OF PROGRAM IN TEACHING

The utility of the program lies in its potential to solve problems governed by equations (4) to (6) with complex boundaries. The utility of the program for learning by the students comes from the fact that it will allow them to try many 'what happens when' questions and will provide them with a better understanding of the heat transfer.

```
Quenching of cylinder 2.0 in diameter and 2.5 in long without grid generation

0,0,10,0                                                       Main parameter card
40,30,7,1                                   Control parameters for element data
2.0833333,2.0833333,1.041666667,200.0,0.0311921           Material property set
 1, 1, 7, 6,1,0.00,0.00,0.25,0.50,0.00,0.50                        Element data
 2, 1, 2, 7,1,0.00,0.00,0.25,0.00,0.25,0.50
 3, 2, 3, 7,1,0.25,0.00,0.50,0.00,0.25,0.50

Element data for elements 4 to 38

39,29,24,25,1,0.75,2.50,0.75,2.00,1.00,2.00
40,29,25,30,1,0.75,2.50,1.00,2.00,1.00,2.50
180,10,0,0.0002777776                                  Transient parameter card
 2,1,0,0.,0.,0.,0.                                        Convection, fluxes and
 3,1,0,0.,0.,0.,0.                                          heat generation data
 6,1,0,0.,0.,0.,0.
 7,1,0,0.,0.,0.,0.
 8,1,0,0.,0.,0.,0.
 9,2,0,0.,0.,0.,0.
24,1,0,0.,0.,0.,0.
25,2,0,0.,0.,0.,0.
33,2,0,0.,0.,0.,0.
36,2,0,0.,0.,0.,0.
38,2,0,0.,0.,0.,0.
40,2,3,0.,0.,0.,0.
 0,0,0,0.,0.,0.,0.                                             Termination card
1200.0,1200.0,1200.0,1200.0,1200.0,1200.0,                 Initial temperatures
1200.0,1200.0,1200.0,1200.0,1200.0,1200.0,
1200.0,1200.0,1200.0,1200.0,1200.0,1200.0,
1200.0,1200.0,1200.0,1200.0,1200.0,1200.0,
1200.0,1200.0,1200.0,1200.0,1200.0,1200.0.
0,0,0,0,0,0,0,0,0,0                                            Fixed nodal data
```

Fig. 2. Input data for quenching problem without grid generation.

```
QUENCHING OF CYLINDER

MATERIAL PROPERTIES

SET    MAT PROP 1     MAT PROP 2     MAT PROP 3     MAT PROP 4     MAT PROP 5
          KRR            KZZ             H             TINF           CRHO
  1    0.3472E-01     0.3472E-01     0.1734E-01     0.20003+03     0.3119E-01

ELEMENT DATA
                       MATERIAL
 NEL   NODE NUMBERS    SET      R(1)      Z(1)      R(2)      Z(2)      R(3)      Z(3)
  1    1    7    6     1        0.0       0.0       0.2500    0.5000    0.0       0.5000

 ELEMENT DATA OF ELEMENT NUMBERS 2 TO 39 ARE OUTPUTTED HERE

 40   29   25   30     1        0.7500    2.5000    1.0000    2.0000    1.0000    2.5000

TRANSIENT ANALYSIS

     NO. ITERATION 500    TIME INCREMENT 0.01667

TIME =  0.0
  1   0.120000E+04     2   0.120000E+04     3   0.120000E+04     4   0.120000E+04
  5   0.120000E+04     6   0.120000E+04     7   0.120000E+04     8   0.120000E+04
  9   0.120000E+04    10   0.120000E+04    11   0.120000E+04    12   0.120000E+04
 13   0.120000E+04    14   0.120000E+04    15   0.120000E+04    16   0.120000E+04
 17   0.120000E+04    18   0.120000E+04    19   0.120000E+04    20   0.120000E+04
 21   0.120000E+04    22   0.120000E+04    23   0.120000E+04    24   0.120000E+04
 25   0.120000E+04    26   0.120000E+04    27   0.120000E+04    28   0.120000E+04
 29   0.120000E+04    30   0.120000E+04

TIME =  2.6666
  1   0.203177E+03     2   0.203525E+03     3   0.203851E+03     4   0.203705E+03
  5   0.203390E+03     6   0.205390E+03     7   0.205375E+03     8   0.205150E+03
  9   0.204854E+03    10   0.204345E+03    11   0.206277E+03    12   0.206100E+03
 13   0.205881E+03    14   0.205431E+03    15   0.204922E+03    16   0.206258E+03
 17   0.206147E+03    18   0.205841E+03    19   0.205473E+03    20   0.204893E+03
 21   0.205438E+03    22   0.205334E+03    23   0.205193E+03    24   0.204821E+03
 25   0.204375E+03    26   0.202330E+03    27   0.203689E+03    28   0.203781E+03
 29   0.203750E+03    30   0.203362E+03
```

Fig. 3. Partial output showing nodal temperatures at time 2.7 min.

Students can try solutions for different boundary conditions, for different isotropic and anisotropic materials by changing thermal conductivity values and for different wind velocities by changing convective heat transfer coefficients. As an example, the problem described in the previous section can be changed slightly as follows. 'A solid mild steel, 2 in-diameter by 2.5 in long cylinder, initially at 1200°F, is exposed instantaneously to a surface temperature of 200°F and is maintained at 200°F for a long time. How long will it take to reach 200°F at the axial and radial centre of the cylinder?'

The time predicted for this problem using the program was 1 min which compared well with the time predicted by following the procedure of Fishenden and Saunders [6].

REFERENCES

1. F. Krieth, *Principles of heat transfer*. Intext Education Publishers, New York, NY. 656 pages (1973).
2. L. J. Segerlind, *Applied finite element analysis*. John Wiley, New York, NY. 422 pages (1976).
3. L. J. Segerlind, *Applied finite element analysis*, 2nd edition. John Wiley, New York, NY. 427 pages (1984).
4. R. J. Gustafson, Finite element method package user's manual. University of Minnesota, St. Paul, MN. (1977).
5. D. R. Pitts and L. E. Sissom, *Heat transfer*, pp. 74–106. McGraw Hill, New York, NY. (1977).
6. M. Fishenden and O. A. Saunders, The calculation of heat transmission. H.M. Stationery Office, London, pp. 75. (1932).