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Kinematic Considerations in the Classical Analysis of Shells of Revolution by Reference to the Geckeler Approximation*

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In this article, a two-stage procedure for computing the total-displacement components throughout an axisymmetric spherical shell is outlined on the basis of the Geckeler approximation. While the approach—which regards the membrane solution as a good approximation to the particular solution of the general shell equations, and hence simply superimposes this solution and the (homogeneous) edge effect—is adequately described in the literature with regard to stress computations, there appear to be obscurities in its application to displacement computations. In particular, it is not clear how the integration constants associated with membrane or bending-disturbance displacements may be determined; furthermore, one often finds that students tacitly assume that, while the two-stage classical bending-theory process (such as Geckeler's approach) gives an accurate—yet simple—way of obtaining stresses in a spherical shell, it is somehow unsuitable for estimating displacements. In this paper, the notion of separate integration constants is clarified and the apparent unsuitability of the classical scheme for deflection computations removed by fully addressing the whole question of kinematic boundary conditions (and the role they play in the overall scheme of stress and displacement computations). Various support conditions for open and closed shells are covered, and examples presented.

NOTATION

- a shell radius
- b finite displacement at a boundary
- C one of two constants of integration for transverse shear force Q_ϕ
- E Young's modulus of elasticity
- H horizontal force per unit circumferential length
- k constant of integration for displacements
- M moment per unit length (= 'bending moment' in shell)
- N in-plane force per unit length (= 'stress resultant' in shell)
- p pressure normal to shell surface
- Q transverse shear force per unit length (= 'shear' in shell)
- S normal force per unit circumferential length
- t thickness of shell
- v displacement along shell meridian
- V meridional rotation (= 'change in slope') of shell upon deformation
- w displacement normal to shell surface
- W movement of shell normal to its surface
- y vertical component of displacements (i.e. in direction of axis of revolution)
- β one of two constants of integration for transverse shear force Q_ϕ
- δ horizontal component of displacements (i.e. in direction perpendicular to axis of revolution)
- ϵ direct strain
- λ shell slenderness parameter ($= [3(1-\nu^2)(a/t)^2]^{1/4}$)
- ν Poisson's ratio
- σ direct stress
- ϕ meridional angle measured from normal to shell at apex to normal at point in question
- ψ meridional angle measured from normal to shell at support to normal at point in question ($= \phi_s - \phi$).

Superscripts

- b denotes bending-disturbance variables
- m denotes membrane-hypothesis variables
- T denotes 'total'

Subscripts

- b refers to 'bottom' (i.e. lower) portion of shell or its edge
- s refers to support location
- t refers to 'top' (i.e. upper) portion of shell or its edge
- θ denotes variables along hoop direction
- ϕ denotes variables along meridional direction.

Additional notation (for Tables 1–3)

- (i) refers to inner surface of shell
- (o) refers to outer surface of shell.

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INTRODUCTION

As is well known, the stresses in an axisymmetrically-loaded non-shallow spherical shell of constant thickness can be estimated reasonably accurately by means of a two-stage computational procedure involving, initially, the calculation of membrane stresses on the basis of the membrane hypothesis, followed by a determination from compatibility considerations of the bending actions required along the edge of the shell to correct the kinematic inconsistencies in the membrane hypothesis at that edge; the quantification of the bending-disturbance stresses associated with this second stage is almost invariably carried out by means of the Geckeler approximation [1]. Superposition of the membrane and bending-disturbance stresses would then complete the stress calculations. Clearly, it should be possible to adopt a procedure analogous to the two-stage stress analysis to obtain displacements at any point on the shell; yet, surprisingly, it appears that such an equally straightforward procedure for estimating displacements arising from the combined effect of membrane action and the corrective edge actions has not been presented anywhere in the literature. By 'literature' we are not only thinking here in terms of research publications, but also of basic student texts, and it is primarily in this area of teaching of classical shell theory that a gap seems to exist when students are faced with the problem of computing deflexions—as opposed to mere stresses—in accordance with the formal bending theory of shells. Such a shortcoming also manifests itself when one wishes to illustrate to students the relative effect—sometimes quite marked—various different boundary conditions have on the magnitude of displacements.

With regard to the computation of displacements on the basis of the two-stage analysis outlined above, a number of obstacles currently exist. First, there is the need to derive expressions for bending-related displacements explicitly in terms of the transverse-shear variable Q_ϕ, and subsequently in terms of the corrective actions required along the edge. In order to retain consistency with respect to the accuracy of the Geckeler approximation, such expressions would neglect lower-order differential terms in Q_ϕ and V (meridional-rotation variable) in relation to higher-order ones. These expressions do not appear to be available in the literature, unlike those for stress resultants (N_ϕ^b, N_θ^b), bending moments (M_ϕ, M_θ) and deformations (V^b, δ^b), which can be found in any of the widely-used texts covering the bending of shells of revolution—see, for example, Timoshenko and Woinowsky-Krieger [2], pp. 549–551, or Flügge [3], pp. 341–344. Then there is also the necessity of evaluating the constants of integration associated with the expressions for displacements. While it is implied or even stated in a number of sources—these will be discussed later—that only a limited choice of boundary conditions exists for the evaluation of the membrane-displacement constant of integration, it is not clear from the existing literature whether this constant may be invariant for different types of physical boundary conditions (e.g. encastré, hinged, or roller supports), and if so, how the bending-displacement constant of integration ought to be evaluated in order to reflect these obvious differences in the kinematic constraints imposed at the edges of the shell.

To enable a better appreciation of some of these obscurities in classical-shell calculations of deformed shapes, the well-known form of the expressions for displacements in terms of stress resultants is briefly derived here before concluding the current discussion on the background to the present work. For an axisymmetrically-loaded non-shallow thin spherical shell of constant thickness t and radius a, the strains ϵ_ϕ (in the meridional direction) and ϵ_θ (in the hoop direction) are related to the displacements v (in the direction of the tangent to the meridian) and w (in the direction of the normal to the middle surface)—see Fig. 1 for the positive directions of these displacements and their horizontal and vertical components δ, y respectively—in the following manner:

$$\epsilon_\phi = \frac{1}{a}\left(\frac{dv}{d\phi} - w\right) \tag{1a}$$

$$\epsilon_\theta = \frac{1}{a}(v \cot\phi - w). \tag{1b}$$

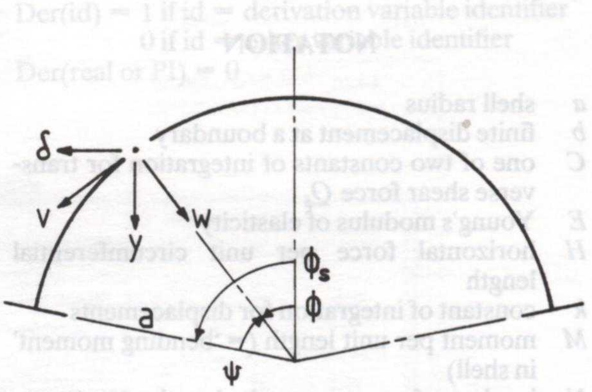

Fig. 1. Positive sign convention for displacements and meridional-angle alternative coordinates for an open spherical shell.

Using the Hookean relations between these strains and the stress resultants, namely

$$\epsilon_\phi = \frac{1}{Et}(N_\phi - \nu N_\theta) \tag{2a}$$

$$\epsilon_\theta = \frac{1}{Et}(N_\theta - \nu N_\phi) \tag{2b}$$

and eliminating w, one obtains a first-order linear differential equation in v, the general solution of which is

$$v = \sin\phi \left[\frac{a}{Et}(1+\nu)\int\left(\frac{N_\phi - N_\theta}{\sin\phi}\right)\mathrm{d}\phi + k\right]. \quad (3a)$$

The normal displacement follows from equations (1b) and (2b):

$$w = v\cot\phi - \frac{a}{Et}(N_\theta - \nu N_\phi). \quad (3b)$$

At this stage, most textbooks simply state that the constant of integration k may be obtained from a kinematic boundary condition (see, for instance, reference [2], p. 447, or reference [3], p. 93). In reference [2], the example of an ideally-supported dome (i.e. with rollers directed so as to give rise to reactions tangential to the surface), for which a membrane solution is known to suffice throughout the entire shell, is considered and k obtained on the basis that $v = 0$ at the support. In this, as in reference [3], it is not clear how one would proceed in the case of non-ideal (i.e. non-tangential) supports. In considering axisymmetric problems for shells of revolution, Kraus [4], pp. 104–105, points out that the constant of integration for the membrane part may not be obtained from a condition such as $w = 0$, but only from a specification for v. Similarly, Novozhilov [5], p. 111, states that only v may be specified for such shells, to avoid violating the assumptions inherent in the membrane hypothesis. However, all these authors omit the question of the calculation of k in the context of bending-disturbance displacements, which must be present in all instances other than the usually impractical case of ideal tangential supports.

It is the aim of this article to reconsider the whole question of kinematic boundary conditions at the support, particularly with regard to how these conditions affect the computation of membrane-solution and bending-solution displacements by the well-established and universally-used two-stage approach. Having derived the relevant expressions on the basis of open (single-domain) shells, the proposed approach for displacement computations is applied to various support conditions, and extended to cover closed (two-domain) shells with variable support conditions. The suggested procedure is illustrated through examples which include a comparison of the various analytical results with those obtained from a finite-element program. It might be argued that the latter, numerical scheme yields both stresses and displacements automatically and that, therefore, the present formal extension of the classical approach to also cater for shell displacements is not essential in view of current computational trends. However, two counter-arguments may be put forward against such criticism. First, the analytical technique still provides the best means for conducting quick parametric studies at sensible costs. Secondly, and more fundamentally, the two-stage process of classical shell calculations retains its didactic advantages in readily linking shell theory to more basic calculations on simpler indeterminate structures, in illustrating the nature of certain approximations and, for the case of displacement computations, in clarifying the kinematics of both 'membrane' and 'bending' effects. The present paper, therefore, should be of particular interest to teachers of shell theory at both undergraduate and postgraduate levels. Although the examples considered were aimed specifically at civil- and mechanical-engineering students (domes, tanks), the notions are equally applicable to the more general structural field, thus encompassing, for example, aeronautical-engineering and naval-architecture studies. The material can be covered in about three lecture periods, which may conveniently be given at the end of the standard exposition of Geckeler's method for stress analysis.

MEMBRANE DISPLACEMENTS

Equations (3) can be written as

$$v^m = \sin\phi\left[\frac{a}{Et}(1+\nu)\int\left(\frac{N_\phi^m - N_\theta^m}{\sin\phi}\right)\mathrm{d}\phi + k^m\right] \quad (4a)$$

$$w^m = v^m\cot\phi - \frac{a}{Et}(N_\theta^m - \nu N_\phi^m)$$
$$= \cos\phi\left[\frac{a}{Et}(1+\nu)\int\left(\frac{N_\phi^m - N_\theta^m}{\sin\phi}\right)\mathrm{d}\phi + k^m\right]$$
$$-\frac{a}{Et}(N_\theta^m - \nu N_\phi^m) \quad (4b)$$

where the superscript m denotes membrane-solution variables. Clearly, the stress resultants N_ϕ^m and N_θ^m, and hence also the integral in the above expressions, depend on the particular loading over the surface of the shell.

In order to maintain consistency with the membrane-hypothesis assumptions at a boundary of an axisymmetrically-loaded shell of revolution [4,5], the constant of integration k^m must be evaluated using the condition

$$v_s^m = b \quad (5)$$

where the subscript s denotes quantities at the support, and b is a finite displacement which will be taken as zero in the present study since the supports are assumed to be non-deformable (non-zero values, which may apply when the shell rests on an elastic foundation, or if it is attached to another deformable structure, would not alter the generality of the approach presently adopted). It must be noted here that the membrane hypothesis permits the specification of only those boundary displacements which occur in directions tangential to the shell midsurface (thus, in the present context, only v^m may be specified at a boundary, although in the more general circumstances of non-axisymmetric loading, the displacment tangential to a circle of latitude may also be specified). Just as the 'membrane' shell cannot resist out-of-plane (boundary) forces, it cannot admit out-of-plane kinematic boundary constraints, as these would imply non-tangential reactions.

BENDING-DISTURBANCE DISPLACEMENTS

Displacements v^b^ *and* w^b^ *in terms of transverse shear* Q_ϕ

Stress resultants N_ϕ^b and N_θ^b are related to the transverse shear force Q_ϕ as follows (reference [2], p. 550):

$$N_\phi^b = -Q_\phi \cot\phi \tag{6a}$$

$$N_\theta^b = -\frac{\mathrm{d}Q_\phi}{\mathrm{d}\phi} \tag{6b}$$

(The superscript b denotes bending-disturbance variables.) To obtain an expression for v^b in terms of Q_ϕ, we substitute N_ϕ^b and N_θ^b as given by equations (6) into equation (3a). This gives

$$v^b = \sin\phi \left[\frac{a}{Et}(1+\nu)\int\left(-Q_\phi \frac{\cos\phi}{\sin^2\phi} + \frac{\mathrm{d}Q_\phi}{\mathrm{d}\phi}\frac{1}{\sin\phi}\right)\mathrm{d}\phi + k^b\right]. \tag{7a}$$

Now, we note that

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{Q_\phi}{\sin\phi}\right) = -Q_\phi\frac{\cos\phi}{\sin^2\phi} + \frac{\mathrm{d}Q_\phi}{\mathrm{d}\phi}\frac{1}{\sin\phi} \tag{7b}$$

and, integrating both sides of this expression with respect to ϕ, yields

$$\frac{Q_\phi}{\sin\phi} = \int\left(-Q_\phi\frac{\cos\phi}{\sin^2\phi} + \frac{\mathrm{d}Q_\phi}{\mathrm{d}\phi}\frac{1}{\sin\phi}\right)\mathrm{d}\phi \tag{7c}$$

the right-hand side of which is precisely the integral in equation (7a). Thus equation (7a) becomes simply

$$v^b = \sin\phi\left[\frac{a}{Et}(1+\nu)\frac{Q_\phi}{\sin\phi} + k^b\right] = \frac{a}{Et}(1+\nu)Q_\phi + k^b\sin\phi \tag{7d}$$

and w^b follows from relation (3b):

$$w^b = v^b\cot\phi - \frac{a}{Et}(N_\theta^b - \nu N_\phi^b). \tag{7e}$$

It is easy to show that $N_\phi^b \ll N_\theta^b$ (see, for example, reference [6], but the proof is evident by inspection of relations (6) and the exponential form of Q_ϕ given by (8) below, noting, at the same time, that λ, the slenderness ratio, is usually not less than about 10). Since ν is also considerably smaller than unity, νN_ϕ^b will be very small in comparison to N_θ^b. Thus, consistent with the accuracy of the Geckeler approximation, the term νN_ϕ^b is dropped out from equation (7e), which becomes

$$w^b = v^b\cot\phi - \frac{a}{Et}N_\theta^b. \tag{7f}$$

(Note: the N_ϕ^b term in relation (3a), that leads to (7a), is retained in the latter expression since it allows a neat expression to be formed, which is amenable to direct integration.)

Displacements v^b^ *and* w^b^ *in terms of edge corrections* M_s *and* H_s

For a shell with one edge, it is convenient to express Q_ϕ in the form

$$Q_\phi = C e^{-\lambda\psi}\sin(\lambda\psi + \beta) \tag{8}$$

(see, for example, reference [2], p. 550), where C and β are constants of integration, and ψ $(=\phi_s - \phi)$ is an alternative coordinate system, as shown in Fig. 1. Then, simple consideration of static boundary conditions for the shell loaded only along the one edge by a uniformly-distributed moment M_s per unit length gives the following solutions for the constants β and C:

$$\beta = 0 \tag{9a}$$

$$C = \frac{2\lambda M_s}{a} \tag{9b}$$

If instead of M_s, a uniformly-distributed horizontal force H_s per unit length is applied along the edge of the otherwise unloaded shell, the constants become

$$\beta = -\frac{\pi}{4} \tag{9c}$$

$$\mathrm{y}C = -\sqrt{2}\,H_s\sin\phi_s \tag{9d}$$

where ϕ_s is the value of ϕ at the support. Details of the evaluation of β and C for either M_s or H_s are readily available (see, for example, reference [2], pp. 550–551). The positive sign convention for M_s and H_s may be seen in Fig. 2(a).

Substituting the results for the relevant β and C (equations (9)) into equation (8) gives Q_ϕ in terms of the superimposed effects of M_s and H_s, enabling Q_ϕ to be eliminated from equation (7d), thus yielding an expression for v^b in terms of M_s and H_s:

$$v^b = \frac{1}{Et}(1+\nu)e^{-\lambda\psi}[2\lambda M_s(\sin\lambda\psi) - aH_s\sin\phi_s(\sin\lambda\psi - \cos\lambda\psi)] + k^b\sin(\phi_s - \psi). \tag{10a}$$

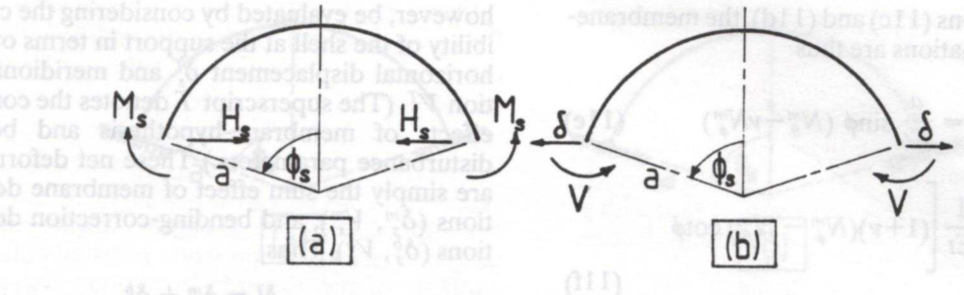

Fig. 2. Positive sign convention for an open spherical shell: (a) shell-edge corrective actions; (b) shell-edge deformations.

The displacement w^b follows from equation (7f), in which N_θ^b may also be expressed in terms of M_s and H_s through successive substitutions in equations (6b), (8) and (9). The result is

$$w^b = v^b \cot(\phi_s - \psi) + \frac{2\lambda}{Et} e^{-\lambda\psi}[\lambda M_s(\sin\lambda\psi - \cos\lambda\psi) + (aH_s \sin\phi_s)\cos\lambda\psi]. \quad (10b)$$

At the support location (where $\phi = \phi_s$, $\psi = 0$), v^b and w^b reduce to

$$v_s^b = \left[\frac{1}{Et}(1+\nu)aH_s + k^b\right]\sin\phi_s \quad (10c)$$

$$w_s^b = \left[\frac{1}{Et}(1+\nu)aH_s + k^b\right]\cos\phi_s - \frac{2\lambda}{Et}[\lambda M_s - aH_s\sin\phi_s]. \quad (10d)$$

Before discussing the procedure for calculating the constant of integration k^b, the computation of the deformations δ and V (required for the purposes of evaluating M_s and H_s) is reviewed.

DEFORMATIONS: RELATION TO DISPLACEMENTS AND STRESS RESULTANTS

The deformations δ (the horizontal component of the displacements v and w) and V (the rotation of the meridian) may be defined as follows:

$$\delta = v\cos\phi - w\sin\phi \quad (11a)$$

$$V = \frac{1}{a}\left(v + \frac{dw}{d\phi}\right). \quad (11b)$$

The positive sign convention for these variables is defined in Fig. 2(b). Let us substitute the expressions for v and w (equations (3)) into the above two equations. The first equation becomes

$$\delta = \sin\phi\cos\phi\left[\frac{a}{Et}(1+\nu)\int\left(\frac{N_\phi - N_\theta}{\sin\phi}\right)d\phi + k\right] - \cos\phi\sin\phi\left[\frac{a}{Et}(1+\nu)\int\left(\frac{N_\phi - N_\theta}{\sin\phi}\right)d\phi + k\right] + \frac{a}{Et}\sin\phi(N_\theta - \nu N_\phi)$$

that is,

$$\delta = \frac{a}{Et}\sin\phi\,(N_\theta - \nu N_\phi) \quad (11c)$$

while the second assumes the form

$$V = \frac{1}{a}\left(\sin\phi\left[\frac{a}{Et}(1+\nu)\int\left(\frac{N_\phi - N_\theta}{\sin\phi}\right)d\phi + k\right] - \sin\phi\left[\frac{a}{Et}(1+\nu)\int\left(\frac{N_\phi - N_\theta}{\sin\phi}\right)d\phi + k\right] + \cos\phi\left[\frac{a}{Et}(1+\nu)\left(\frac{N_\phi - N_\theta}{\sin\phi}\right)\right] - \frac{a}{Et}\frac{d}{d\phi}(N_\theta - \nu N_\phi)\right)$$

that is,

$$V = \frac{1}{Et}\left[(1+\nu)(N_\phi - N_\theta)\cot\phi - \frac{d}{d\phi}(N_\theta - \nu N_\phi)\right] \quad (11d)$$

Expressions (11c) and (11d) are clearly independent of k. This means that the deformations δ and V need not be determined via the displacements v and w (expressions (11a) and (11b)), as they may be calculated directly (from (11c) and (11d)) once the stress resultants N_ϕ and N_θ are known. This is true for both membrane and bending-disturbance variables (note that variables in (11) are unsuperscripted, as they are equally valid for both calculation stages). While v and w are individually dependent upon k^m or k^b as appropriate, the definitions of δ and V in terms of v and w are such that terms in k^m or k^b vanish, as has just been shown, to leave only terms in the stress resultants N_ϕ and N_θ.

From equations (11c) and (11d), the membrane-solution deformations are thus

$$\delta^m = \frac{a}{Et}\sin\phi\,(N_\theta^m - \nu N_\phi^m) \tag{11e}$$

$$V^m = \frac{1}{Et}\left[(1+\nu)(N_\phi^m - N_\theta^m)\cot\phi - \frac{d}{d\phi}(N_\theta^m - \nu N_\phi^m)\right]. \tag{11f}$$

The expressions for the bending-disturbance deformations have the same form (with the superscript b now replacing m), but since N_ϕ^b and N_θ^b are known in terms of M_s and H_s, δ^b and V^b are more usefully expressed in the form

$$\delta^b = -\frac{2\lambda}{Et}\sin(\phi_s - \psi)\,e^{-\lambda\psi}[\lambda M_s \sin\lambda\psi - (\lambda M_s - aH_s\sin\phi_s)\cos\lambda\psi] \tag{11g}$$

$$V^b = -\frac{2\lambda^2}{Eat}e^{-\lambda\psi}[aH_s\sin\phi_s(\sin\lambda\psi - \cos\lambda\psi) + 2\lambda M_s\cos\lambda\psi]. \tag{11h}$$

In deriving the above two equations, it should be remembered that νN_ϕ^b must be neglected in comparison to N_θ^b, as should any other low-order differential terms in V^b or Q_ϕ in relation to higher-order ones. Neglecting such low-order terms is not only a good approximation, but also necessary for consistency (inconsistencies in the accuracy of various stages of the present procedure may lead to difficulties later on when it comes to enforcing boundary conditions, where it may prove impossible to satisfy all the required kinematic boundary conditions simultaneously). Throughout this study, an accuracy consistent with that of the Geckeler approximation is maintained.

Of interest to us will be the values of δ^b and V^b at the support. Their expressions are:

$$\delta_s^b = \frac{2\lambda}{Et}\sin\phi_s\,(\lambda M_s - aH_s\sin\phi_s) \tag{11i}$$

$$V_s^b = -\frac{2\lambda^2}{Eat}(2\lambda M_s - aH_s\sin\phi_s). \tag{11j}$$

ROLE OF BOUNDARY CONDITIONS IN THE DETERMINATION OF BENDING-DISTURBANCE STRESSES AND DISPLACEMENTS

Up to this stage the corrective actions M_s and H_s have been given in an arbitrary manner. They can, however, be evaluated by considering the compatibility of the shell at the support in terms of its net horizontal displacement δ_s^T and meridional rotation V_s^T. (The superscript T denotes the combined effects of membrane-hypothesis and bending-disturbance parameters.) These net deformations are simply the sum effect of membrane deformations (δ_s^m, V_s^m), and bending-correction deformations (δ_s^b, V_s^b). Thus

$$\delta_s^T = \delta_s^m + \delta_s^b \tag{12a}$$

$$V_s^T = V_s^m + V_s^b. \tag{12b}$$

It has already been shown that δ_s^m and V_s^m are independent of k^m, while δ_s^b and V_s^b are independent of k^b. In fact, an examination of relations (3) reveals that the constants k^m and k^b merely represent rigid-body (hence strain-free) displacements vertically downwards (i.e. in the direction of the axis of revolution). Since k^m and k^b are associated with stress-free terms, there is no need to enforce compatibility of the shell at the support in the vertical direction if the purpose of the analysis is merely to obtain stresses. In other words, expressions (12) are generally sufficient for the purposes of obtaining stresses. However, as will be seen later, additional conditions effectively enforcing compatibility in the vertical direction will be required to evaluate k^b (k^m is evaluated on the basis of the invariant condition $v_s^m = 0$, as has already been explained) if the analysis also requires a full definition of displacements. This is not clear in existing textbooks, in which analysis is not carried out beyond stress computations.

Let us consider the various limiting-support cases shown in Fig. 3, where (a) represents a shell with encastré supports, (b) a shell with fully-pinned (i.e. hinged) supports, (c) one with pinned supports resting on vertically-reacting rollers, and (d) the same as (c) but with the reactions now tangential to the shell rather than vertical. For case (a), the following conditions apply at the support:

$$\delta_s^T = 0 \tag{13a}$$

$$V_s^T = 0. \tag{13b}$$

Solution of equations (13a) and (13b) enables the corrective actions M_s and H_s to be evaluated for the encastré support. In the case of the hinged support (Fig. 3(b)), we specify only condition (13a) (since $M_s = 0$) and solve for H_s. With regard to Fig. 3(c), M_s is again zero (because of the moment-release afforded by the pin in the support), while H_s can also be determined purely from statics (H_s, being directed towards the axis of revolution of the shell, is equal in magnitude but opposite in direction to the horizontal component of the positive (i.e. tensile) membrane meridional stress resultant at the support). When the pinned edge is provided with tangentially-reacting (instead of vertically-reacting) rollers (Fig. 3(d)), not only M_s but also H_s is zero from statics. This last case represents the ideal membrane boundary conditions that permit a

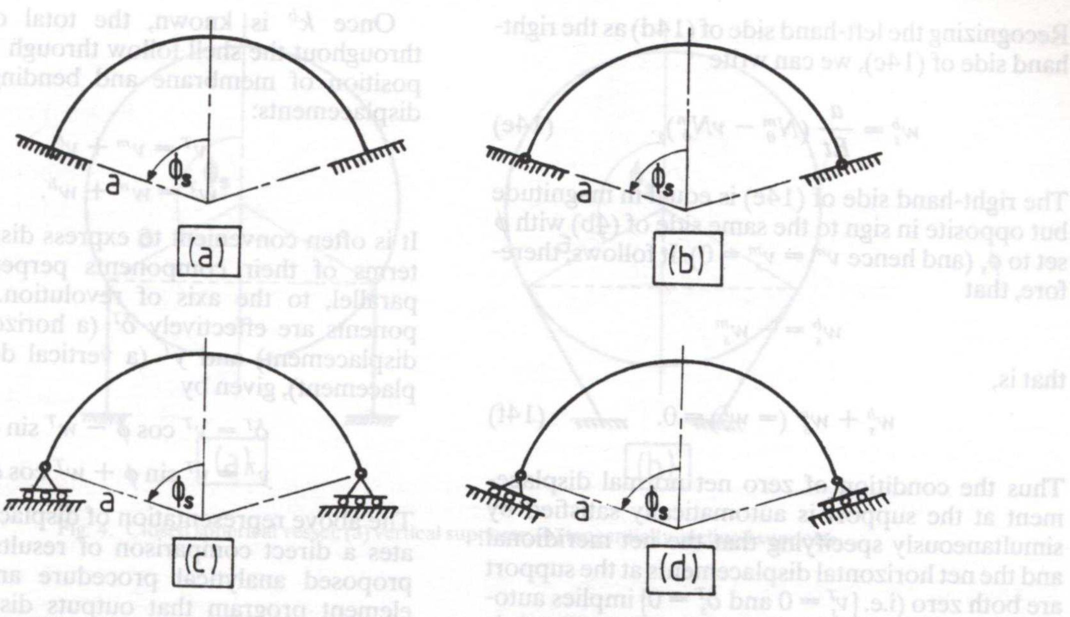

Fig. 3. Boundary conditions for an open spherical shell: (a) encastré support; (b) fully-pinned support; (c) pinned support on rollers—vertical reactions; (d) pinned support on rollers—tangential reactions.

purely membrane state of stress to exist throughout the entire shell.

Once M_s and H_s have been evaluated, the bending-disturbance stresses throughout the support zone readily follow (see, for example, reference [2], p. 550, for the relevant expressions), and when these are combined with the membrane stresses, the state of stress in the whole shell becomes fully defined. The shell problem may then be regarded as completely solved unless, in addition to the stresses, displacements are also required.

We are now in a position to evaluate k^b, the constant of integration associated with the bending-disturbance displacements v^b and w^b. For this purpose, we will refocus attention on the various support cases depicted in Fig. 3.

With reference to the encastré support (Fig. 3(a)) and the hinged one (Fig. 3(b)), it is quite clear that the net meridional displacement at the support ought to be zero. Thus

$$v_s^T = v_s^m + v_s^b = 0. \qquad (14a)$$

Since $v_s^m = 0$ (an invariant membrane boundary condition), it follows that $v_s^b = 0$. Equating the right-hand side of equation (10c) to zero, we obtain

$$k^b = -\frac{1}{Et}(1+\nu)aH_s. \qquad (14b)$$

Expression (14b), when substituted into (10a), enables the displacement v^b and hence also w^b, to be fully determined at any point on the shell. It is interesting to note that k^b for the encastré support and the fully-pinned one is independent of M_s (the flexural corrective action), and not directly dependent on ϕ_s, although it may be indirectly related to the support angle through H_s. Although expression (14b) for k^b applies to both the encastré and the fully-pinned support, its value differs between the two cases because, in general, their H_s values differ. Thus, and as one would expect, the displacements v^b and w^b will be different for the encastré and fully-pinned cases for two reasons—first, the deformation components differ because of M_s and H_s which will not be the same for the two types of support (see expressions (10a) and (10b)); then, the translational (i.e. rigid-body displacement) components also differ since these are in terms of k^b, which, as we have seen, differs for the two types of support in question (since H_s in expression (14b) is not common to both problems). We will now consider other consequences of condition (14a).

Substituting expression (14b) for k^b into (10b) for w^b, we obtain, for $\psi = 0$,

$$w_s^b = -\frac{2\lambda}{Et}(\lambda M_s \quad aH_s\sin\phi_s). \qquad (14c)$$

It may be recalled that the condition $\delta_s^T (= \delta_s^m + \delta_s^b) = 0$ has already been used in the evaluation of the edge corrective actions applicable to support cases in Fig. 3(a) and Fig. 3(b) (see equation (13a)). If we now expand the terms δ_s^m (using (11e) for $\phi = \phi_s$) and δ_s^b (using (11i)), this condition (i.e. (13a)) becomes

$$\frac{a}{Et}\sin\phi_s(N_\theta^m - \nu N_\phi^m)_s + \frac{2\lambda}{Et}\sin\phi_s(\lambda M_s - aH_s\sin\phi_s) = 0$$

that is,

$$\frac{2\lambda}{Et}(\lambda M_s - aH_s\sin\phi_s) = \frac{a}{Et}(N_\theta^m - \nu N_\phi^m)_s. \qquad (14d)$$

Recognizing the left-hand side of (14d) as the right-hand side of (14c), we can write

$$w_s^b = \frac{a}{Et}(N_\theta^m - \nu N_\phi^m)_s. \quad (14e)$$

The right-hand side of (14e) is equal in magnitude but opposite in sign to the same side of (4b) with ϕ set to ϕ_s (and hence $v^m = v_s^m = 0$). It follows, therefore, that

$$w_s^b = -w_s^m$$

that is,

$$w_s^b + w_s^m (= w_s^T) = 0. \quad (14f)$$

Thus the condition of zero net normal displacement at the support is automatically satisfied by simultaneously specifying that the net meridional and the net horizontal displacements at the support are both zero (i.e. $\{v_s^T = 0$ and $\delta_s^T = 0\}$ implies automatically that $w_s^T = 0$). As one would expect, it is also quite clear that the condition of zero net vertical displacement at the support (i.e. $y_s^T = v_s^T \sin\phi_s + w_s^T \cos\phi_s = 0$) is also automatically satisfied since $v_s^T = 0$, and w_s^T has just been shown to be zero (i.e. $\{v_s^T = 0$ and $\delta_s^T = 0\}$ implies automatically that $y_s^T = 0$). (In existing textbooks, where only stress computations are performed, only v_s^m and δ_s^T are discussed, and no formal proof of full total-displacement compatibility at the support seems to have been outlined.) Earlier on, reference has been made to the necessity of maintaining a consistent level of accuracy at all stages. It may be pointed out that even slight deviations from consistency can make it at least difficult (if not impossible) to achieve automatic compatibility of displacements in any direction in the meridional plane at the support just by specifiying displacements in two other non-colinear directions.

Considering now Fig. 3(c), where the support is pinned on rollers permitting horizontal movement, it is clear that we can only enforce the condition of zero net vertical displacement as a way of obtaining k^b:

$$y_s^T = y_s^m + y_s^b = (v_s^m \sin\phi_s + w_s^m \cos\phi_s) + (v_s^b \sin\phi_s + w_s^b \cos\phi_s) = 0. \quad (15a)$$

Now $v_s^m = 0$ (so that $w_s^m = -(a/Et)(N_\theta^m - \nu N_\phi^m)_s$—see expression (3b)). Using, in condition (15a), these values for v_s^m and w_s^m, and the expressions for v_s^b (10c) and w_s^b (10d), leads to the result

$$k^b = \frac{1}{Et}\left[a(N_\theta^m - \nu N_\phi^m)_s + 2\lambda\,(\lambda M_s - aH_s \sin\phi_s)\right] \times \cos\phi_s - \frac{1}{Et}(1+\nu)aH_s. \quad (15b)$$

Finally, for the case depicted in Fig. 3(d), there are no bending disturbances in the shell, so that $v^T = v^m$, $w^T = w^m$, and, clearly, these displacements become fully defined once k^m has been determined from the condition $v_s^m = 0$.

Once k^b is known, the total displacements throughout the shell follow through simple superposition of membrane and bending-disturbance displacements:

$$v^T = v^m + v^b \quad (16a)$$

$$w^T = w^m + w^b. \quad (16b)$$

It is often convenient to express displacements in terms of their components perpendicular, and parallel, to the axis of revolution. These components are effectively δ^T (a horizontal outward displacement) and y^T (a vertical downward displacement), given by

$$\delta^T = v^T \cos\phi - w^T \sin\phi \quad (16c)$$

$$y^T = v^T \sin\phi + w^T \cos\phi. \quad (16d)$$

The above representation of displacements facilitates a direct comparison of results between the proposed analytical procedure and any finite-element program that outputs displacements in global Cartesian directions.

CLOSED SHELLS WITH VERTICAL AND INCLINED SUPPORTS

For a closed spherical shell propped on vertically-reacting supports, such as the liquid-retaining vessel shown in Fig. 4(a), the two-stage procedure that has just been described for single-domain shells may be used to determine the stresses and displacements in the upper and lower domains of the closed shell. For each domain, membrane displacements are computed on the basis that $v_s^m = 0$. This condition allows k_t^m and k_b^m (the subscripts t and b refer to the 'top' (i.e. upper) and 'bottom' (i.e. lower) domains of the shell respectively, the circle of support being the demarcation between the two regions) to be determined. In general, two pairs of corrective actions $\{M_t, H_t\}$, $\{M_b, H_b\}$ and deformation variables $\{\delta_t, V_t\}$, $\{\delta_b, V_b\}$ now apply. The positive sign convention for these variables is shown in Fig. 5(a) (corrective actions) and Fig. 5(b) (deformations). As before, the deformation variables are independent of k_t^m, k_t^b, k_b^m or k_b^b, as appropriate, and may be calculated as soon as the stress resultants in each domain are known. The four corrective actions depend on the stiffness of the circular ring beam at the junction of the two domains. For example, in the case of a fully-rigid ring beam, the four actions are obtained by solving the system of simultaneous equations stemming from the conditions

$$\delta_t^T (= \delta_t^m + \delta_t^b) = 0 \quad (17a)$$

$$\delta_b^T (= \delta_b^m + \delta_b^b) = 0 \quad (17b)$$

$$V_t^T (= V_t^m + V_t^b) = 0 \quad (17c)$$

$$V_b^T (= V_b^m + V_b^b) = 0 \quad (17d)$$

while, for a ring beam that is completely flexible both circumferentially and torsionally, the deformations of the two portions of the shell must be

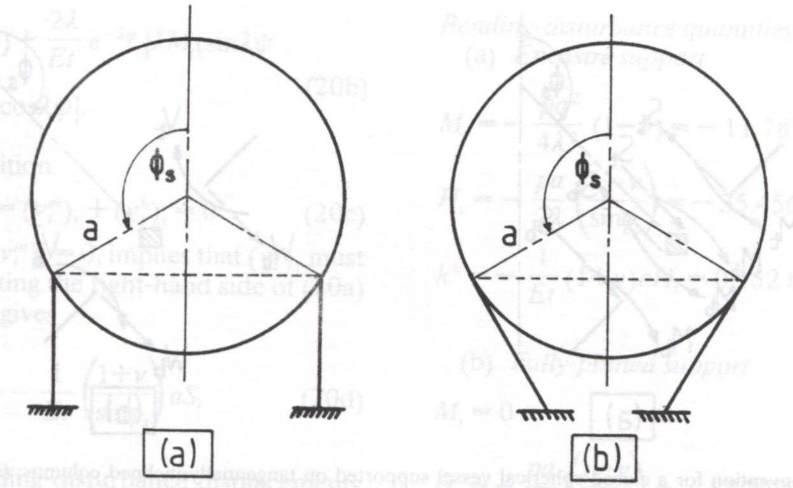

Fig. 4. Closed spherical vessel: (a) vertical supports; (b) tangentially-inclined supports.

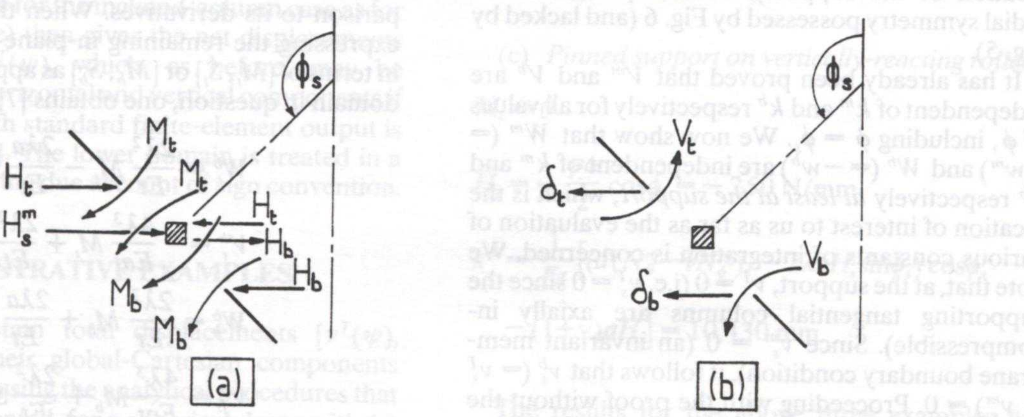

Fig. 5. Positive sign convention for a closed spherical vessel supported on vertical columns: (a) shell-edge corrective actions and ring-beam equilibrium actions; (b) shell-edge deformations.

matched at the support, and, at the same time, equilibrium of the ring beam must be maintained:

$$\delta_t^T (= \delta_t^m + \delta_t^b) = \delta_b^T (= \delta_b^m + \delta_b^b) \quad (17e)$$

$$V_t^T (= V_t^m + V_t^b) = V_b^T (= V_b^m + V_b^b) \quad (17f)$$

$$M_t = M_b \quad (17g)$$

$$H_t = H_b + H_s^m. \quad (17h)$$

In (17h), H_s^m is the horizontal component of the membrane meridional stress resultants at the support, H_s^m being positive when directed towards the axis of revolution (see Fig. 5(a)).

Once M_t, H_t, M_b and H_b are known, the condition of zero net vertical displacements at the support (i.e. $y_s^T = 0$, assuming that the supporting columns are incompressible) should be applied to the upper and lower domains separately to obtain k_t^b and k_b^b respectively. The rest of the procedure to obtain displacements (and stresses, if required) follows that already described for single-domain open shells, taking due account of sign convention for each domain, and using the variables appropriate to the domain in question.

When the closed spherical shell is supported on tangentially-inclined columns, as in Fig. 4(b), a different system of corrective actions and deformations is proposed. Figure 6 depicts the suggested system, with (a) showing corrective actions and (b) the relevant deformations. The difference between the well-known system depicted in Fig. 5 and our adaptation for tangential supports shown in Fig. 6 is that the horizontal corrective actions $\{H_t, H_b\}$ have now been replaced by actions normal to the shell surface $\{S_t, S_b\}$, while displacement deformations normal to the shell surface $\{W_t, W_b\}$ are now of interest instead of the previous horizontally-orientated variables $\{\delta_t, \delta_b\}$. The membrane variable H_s^m, which featured in the previous system, has no counterpart in the new system because membrane actions cannot have a component in the direction normal to the shell midsurface. It may be noted that the corrective forces $\{S_t, S_b\}$ are actually the values of the transverse shear force Q_ϕ at the edges, a feature which makes the two-stage bending analysis of the tangentially-supported vessel considerably simpler than that of the vertically-supported one. As will be seen shortly, the bending-disturbance deformations at the support $\{W_t^b, V_t^b\}$ and $\{W_b^b, V_b^b\}$ are all independent of ϕ_s (the

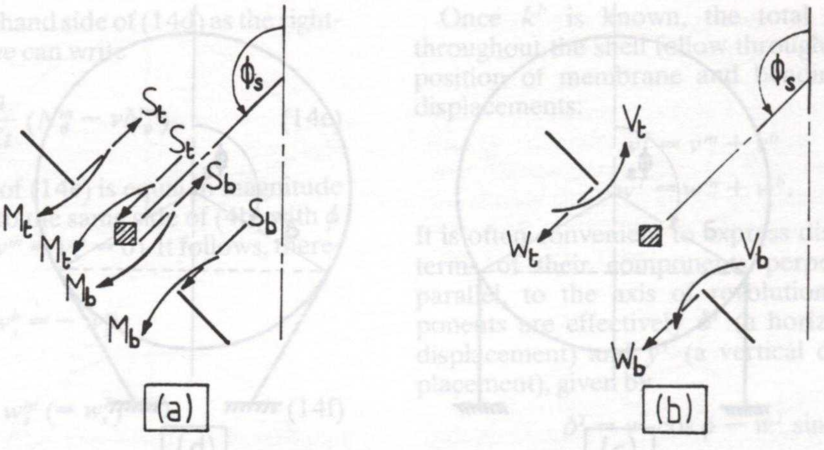

Fig. 6. Positive sign convention for a closed spherical vessel supported on tangentially-inclined columns: (a) shell-edge corrective actions and ring-beam equilibrium actions; (b) shell-edge deformations.

location of the support), a consequence of the radial symmetry possessed by Fig. 6 (and lacked by Fig. 5).

It has already been proved that V^m and V^b are independent of k^m and k^b respectively for all values of ϕ, including $\phi = \phi_s$. We now show that W^m ($= -w^m$) and W^b ($= -w^b$) are independent of k^m and k^b respectively *at least at the support*, which is the location of interest to us as far as the evaluation of various constants of integration is concerned. We note that, at the support, $v^T = 0$ (i.e. $v^T_s = 0$ since the supporting tangential columns are axially incompressible). Since $v^m_s = 0$ (an invariant membrane boundary condition), it follows that v^b_s ($= v^T_s - v^m_s$) $= 0$. Proceeding with the proof without the use of superscripts m and b (the remainder applies equally for membrane as for bending-disturbance variables), we note that δ ($= v \cos \phi - w \sin \phi$) reduces to $\delta_s = -w_s \sin \phi_s$ at the support since $v_s = 0$. Rearranging this relation, $w_s = -\delta_s/\sin \phi_s$. Since δ_s is independent of k (it was proved in earlier sections that δ is independent of k for all values of ϕ), it follows that w_s, and hence W_s ($= -w_s$), is independent of k. This enables us to write down the expressions for W_s (and for V_s, which are as in previous sections) independently of k. Thus the membrane deformations at the support are simply

$$W^m_s = \frac{a}{Et}(N^m_\theta - \nu N^m_\phi)_s \tag{18a}$$

$$V^m_s = \frac{1}{Et}\left((1+\nu)(N^m_\phi - N^m_\theta)_s \cot\phi_s - \left[\frac{\mathrm{d}}{\mathrm{d}\phi}(N^m_\theta - \nu N^m_\phi)\right]_s\right) \tag{18b}$$

where the subscript s may refer to the edge of either the upper domain or the lower one. The bending-disturbance deformations are also given by a pair of expressions of the same form as (18a) and (18b), but with superscript b in place of m. However, as explained earlier, νN^b_ϕ must be neglected in relation to N^b_θ, as must any Q_ϕ-related variable in comparison to its derivatives. When this is done, and expressing the remaining in-plane stress resultant in terms of $\{M_t, S_t\}$ or $\{M_b, S_b\}$ as appropriate for the domain in question, one obtains [7]

$$W^b_t = \frac{2\lambda^2}{Et} M_t - \frac{2\lambda a}{Et} S_t \tag{18c}$$

$$V^b_t = -\frac{4\lambda^3}{Eat} M_t + \frac{2\lambda^2}{Et} S_t \tag{18d}$$

$$W^b_b = \frac{2\lambda^2}{Et} M_b + \frac{2\lambda a}{Et} S_b \tag{18e}$$

$$V^b_b = \frac{4\lambda^3}{Eat} M_b + \frac{2\lambda^2}{Et} S_b \tag{18f}$$

for the bending-disturbance deformations at the support. These expressions are clearly independent of ϕ_s (as already stated), except in so far as M_t, S_t, M_b and S_b may be functions of ϕ_s.

The corrective actions M_t, S_t, M_b and S_b are evaluated in a manner similar to that for vertically-supported vessels. Considering the case of the completely flexible ring beam as an example, the boundary conditions

$$W^T_t (= W^m_t + W^b_t) = W^T_b (= W^m_b + W^b_b) \tag{19a}$$

$$V^T_t (= V^m_t + V^b_t) = V^T_b (= V^m_b + V^b_b) \tag{19b}$$

$$M_t = M_b \tag{19c}$$

$$S_t = S_b \tag{19d}$$

enable all the corrective actions to be determined [7].

The derivation of displacement expressions follows a sequence closely resembling that for single-domain shells. For the upper domain, for example, we simply use expressions (10a) and (10b), with v^b replaced by v^b_t, w^b by w^b_t, k^b by k^b_t, M_s by M_t; also ($H_s \sin \phi_s$) is replaced by S_t [7]. Thus

$$v^b_t = \frac{1}{Et}(1+\nu)e^{-\lambda\psi}[2\lambda M_t(\sin\lambda\psi) - aS_t(\sin\lambda\psi - \cos\lambda\psi)] + k^b_t \sin(\phi_s - \psi) \tag{20a}$$

$$w_t^b = v_t^b \cot(\phi_s - \psi) + \frac{2\lambda}{Et} e^{-\lambda\psi} [\lambda M_t(\sin\lambda\psi - \cos\lambda\psi) + aS_t\cos\lambda\psi]. \quad (20b)$$

Applying the condition

$$(v_t^T)_s = (v_t^m)_s + (v_t^b)_s = 0 \quad (20c)$$

and recalling that $(v_t^m)_s = 0$, implies that $(v_t^b)_s$ must also be zero. Equating the right-hand side of (20a) to zero, for $\psi = 0$, gives

$$k_t^b = -\frac{1}{Et}\left(\frac{1+\nu}{\sin\phi_s}\right) aS_t \quad (20d)$$

Adding the bending-disturbance displacements to the corresponding membrane displacements (which, as in the case of vertical columns, are computed on the basis that $(v_t^m)_s = 0$, and would therefore be the same for the inclined-column case as for the vertical one) then gives the net displacements $v_t^T(\psi)$ and $w_t^T(\psi)$, which, as before, may be resolved into horizontal and vertical components if comparison with standard finite-element output is to be facilitated. The lower domain is treated in a similar way, taking due account of sign convention.

ILLUSTRATIVE EXAMPLES

We now obtain total displacements $\{v^T(\psi), w^T(\psi)\}$, and their global-Cartesian components $\{\delta^T(\psi), y^T(\psi)\}$, using the analytical procedures that have been presented, for a spherical dome with (a) an encastré support, (b) a fully-pinned support, and (c) a pinned support on rollers permitting horizontal movement (see Fig. 3 for these different support conditions). The characteristics relevant to all the three examples are:

ϕ_s = 60°
a = 10 000 mm
t = 100 mm
a/t = 100
ν = 0.20
$\lambda = (3(1-\nu^2)(a/t)^2)^{1/4} = 13.027$
E = 28 000 N/mm²
loading = 0.1 N/mm² (external uniform pressure p).

Thus we have a 10-metre radius dome of a material resembling concrete, and subject to a pressure equivalent to a 10-metre head of water.

Membrane quantities (for all three examples)

$$N_\phi^m = N_\theta^m = -\frac{pa}{2} = -500 \text{ N/mm}$$

$v^m = k^m \sin\phi$

$v_s^m = 0$ implies that $k^m = 0$

Thus $v^m(\psi) = 0$ and $w^m(\psi) = \dfrac{pa^2}{2Et}(1-\nu) = 1.43$ mm.

Bending-disturbance quantities

(a) *Encastré support*

$$M_s = -\frac{pa^2}{4\lambda^2}(1-\nu) = -11\,785 \text{ Nmm/mm}$$

$$H_s = -\frac{pa}{2\lambda}\left(\frac{1-\nu}{\sin\phi_s}\right) = -35.456 \text{ N/mm}$$

$$k^b = -\frac{1}{Et}(1+\nu)aH_s = 0.152 \text{ mm.}$$

(b) *Fully-pinned support*

$$M_s = 0$$

$$H_s = -\frac{pa}{4\lambda}\left(\frac{1-\nu}{\sin\phi_s}\right) = -17.728 \text{ N/mm}$$

$$k^b = -\frac{1}{Et}(1+\nu)aH_s = 0.076 \text{ mm.}$$

(c) *Pinned support on vertically-reacting rollers*

$$M_s = 0$$

$$H_s = -\frac{pa}{2}\cos\phi_s = -250 \text{ N/mm}$$

$$k^b = \frac{1}{Et}\Big[(a(N_\theta^m - \nu N_\phi^m)_s - 2\lambda aH_s\sin\phi_s)\cos\phi_s - (1+\nu)aH_s\Big] = 10.430 \text{ mm.}$$

The results for the above three examples are shown in Tables 1 (encastré support), 2 (fully-pinned support) and 3 (pinned support on rollers permitting horizontal movement). Displacements are in millimetres. The last four columns of each table give total fibre stresses (in N/mm²) for the inner (denoted by (i)) and outer (denoted by (o)) surfaces of the shell respectively, with negative values denoting compression, and positive values tension. In the columns for displacements $\delta^T(\psi)$, $y^T(\psi)$, and stresses $\sigma_\phi^T(\psi)$, $\sigma_\theta^T(\psi)$, first values refer to the analytical results as found from the procedures presented in this article, while second values (in brackets) are those obtained using a finite-element program employing a relatively fine mesh (with elements subtending angles of 1° at the centre).

CONCLUSION

In this article, a step-by-step procedure for computing the displacements in non-shallow spherical shells has been presented. The approach, which is based on the Geckeler approximation, is suitable for shells of constant thickness subjected to axisymmetric loading conditions.

The use of kinematic boundary conditions in the scheme of computations has been elucidated. On

Table 1. Displacements and stresses for the illustrative example—encastré support (Fig. 3(a))

ψ (deg.)	$v^T(\psi)$ (mm)	$w^T(\psi)$ (mm)	$\delta^T(\psi)$ (mm)	$y^T(\psi)$ (mm)	$\sigma_\phi^T(\psi)$(N/mm²)		$\sigma_\theta^T(\psi)$(N/mm²)	
					(i)	(o)	(i)	(o)
0	0.00	0.00	0.00 (0.00)	0.00 (0.00)	−12.07 (−12.07)	+2.07 (+2.42)	−2.41 (−2.41)	+0.41 (+0.48)
1	0.03	0.08	−0.06 (−0.04)	0.07 (0.06)	−9.22 (−9.19)	−0.78 (−0.50)	−2.02 (−1.92)	−0.33 (−0.30)
2	0.05	0.25	−0.18 (−0.16)	0.18 (0.17)	−7.06 (−7.01)	−2.94 (−2.75)	−2.02 (−1.85)	−1.19 (−1.19)
3	0.08	0.46	−0.35 (−0.32)	0.32 (0.30)	−5.52 (−5.46)	−4.48 (−4.38)	−2.26 (−2.05)	−2.05 (−2.06)
4	0.09	0.69	−0.52 (−0.49)	0.46 (0.44)	−4.50 (−4.42)	−5.50 (−5.47)	−2.64 (−2.42)	−2.84 (−2.86)
5	0.11	0.90	−0.67 (−0.65)	0.60 (0.59)	−3.90 (−3.81)	−6.10 (−6.14)	−3.08 (−2.86)	−3.52 (−3.55)
6	0.12	1.08	−0.81 (−0.78)	0.73 (0.72)	−3.60 (−3.51)	−6.40 (−6.48)	−3.51 (−3.32)	−4.07 (−4.11)
7	0.12	1.24	−0.91 (−0.90)	0.84 (0.83)	−3.53 (−3.44)	−6.47 (−6.57)	−3.91 (−3.75)	−4.50 (−4.54)
8	0.12	1.36	−0.99 (−0.98)	0.94 (0.93)	−3.61 (−3.52)	−6.39 (−6.51)	−4.25 (−4.12)	−4.81 (−4.86)
9	0.13	1.45	−1.05 (−1.04)	1.01 (1.01)	−3.77 (−3.69)	−6.23 (−6.35)	−4.53 (−4.43)	−5.02 (−5.07)
10	0.13	1.52	−1.08 (−1.08)	1.07 (1.07)	−3.97 (−3.91)	−6.03 (−6.14)	−4.75 (−4.68)	−5.16 (−5.20)
12	0.12	1.59	−1.10 (−1.10)	1.15 (1.16)	−4.39 (−4.35)	−5.61 (−5.69)	−5.01 (−4.99)	−5.26 (−5.29)
14	0.11	1.60	−1.07 (−1.08)	1.20 (1.20)	−4.72 (−4.70)	−5.28 (−5.33)	−5.12 (−5.13)	−5.23 (−5.25)
16	0.11	1.59	−1.03 (−1.03)	1.22 (1.23)	−4.93 (−4.92)	−5.07 (−5.09)	−5.13 (−5.15)	−5.16 (−5.17)
18	0.10	1.58	−0.98 (−0.98)	1.24 (1.25)	−5.03 (−5.04)	−4.97 (−4.97)	−5.10 (−5.12)	−5.09 (−5.09)
20	0.10	1.56	−0.93 (−0.93)	1.26 (1.27)	−5.06 (−5.07)	−4.94 (−4.93)	−5.06 (−5.08)	−5.04 (−5.04)
40	0.05	1.57	−0.49 (−0.49)	1.50 (1.50)	−5.00 (−5.00)	−5.00 (−5.00)	−5.00 (−5.00)	−5.00 (−5.00)
60	0.00	1.58	0.00 (0.00)	1.58 (1.60)	−5.00 (−5.00)	−5.00 (−5.00)	−5.00 (−5.00)	−5.00 (−5.00)

Table 2. Displacements and stresses for the illustrative example—fully-pinned support (Fig. 3(b))

ψ (deg.)	$v^T(\psi)$ (mm)	$w^T(\psi)$ (mm)	$\delta^T(\psi)$ (mm)	$y^T(\psi)$ (mm)	$\sigma_\phi^T(\psi)$(N/mm²)		$\sigma_\theta^T(\psi)$(N/mm²)	
					(i)	(o)	(i)	(o)
0	0.00	0.00	0.00 (0.00)	0.00 (0.00)	−5.00 (−4.84)	−5.00 (−4.83)	−1.00 (−0.81)	−1.00 (−1.12)
1	0.03	0.34	−0.28 (−0.27)	0.20 (0.19)	−3.73 (−3.66)	−6.27 (−6.24)	−1.64 (−1.45)	−2.15 (−2.26)
2	0.05	0.64	−0.52 (−0.51)	0.38 (0.37)	−3.03 (−2.97)	−6.97 (−6.99)	−2.33 (−2.13)	−3.11 (−3.21)
3	0.06	0.91	−0.73 (−0.71)	0.54 (0.53)	−2.75 (−2.69)	−7.25 (−7.30)	−2.98 (−2.80)	−3.88 (−3.97)
4	0.07	1.12	−0.89 (−0.88)	0.68 (0.68)	−2.75 (−2.70)	−7.25 (−7.33)	−3.56 (−3.40)	−4.46 (−4.53)
5	0.07	1.29	−1.01 (−1.00)	0.80 (0.79)	−2.94 (−2.89)	−7.06 (−7.16)	−4.05 (−3.92)	−4.87 (−4.93)
6	0.07	1.41	−1.10 (−1.09)	0.89 (0.88)	−3.23 (−3.19)	−6.77 (−6.87)	−4.44 (−4.34)	−5.14 (−5.20)
7	0.07	1.49	−1.15 (−1.14)	0.96 (0.96)	−3.56 (−3.52)	−6.44 (−6.54)	−4.73 (−4.66)	−5.30 (−5.35)
8	0.07	1.54	−1.17 (−1.17)	1.01 (1.01)	−3.89 (−3.85)	−6.11 (−6.20)	−4.94 (−4.90)	−5.38 (−5.42)
9	0.07	1.57	−1.18 (−1.18)	1.04 (1.05)	−4.19 (−4.16)	−5.81 (−5.89)	−5.07 (−5.06)	−5.40 (−5.43)
10	0.07	1.58	−1.17 (−1.17)	1.07 (1.07)	−4.44 (−4.43)	−5.56 (−5.62)	−5.15 (−5.16)	−5.38 (−5.40)
12	0.06	1.57	−1.13 (−1.13)	1.10 (1.10)	−4.81 (−4.81)	−5.19 (−5.22)	−5.20 (−5.22)	−5.28 (−5.29)
14	0.06	1.54	−1.07 (−1.07)	1.11 (1.12)	−5.01 (−5.02)	−4.99 (−4.99)	−5.17 (−5.20)	−5.16 (−5.17)
16	0.05	1.52	−1.02 (−1.02)	1.13 (1.13)	−5.09 (−5.10)	−4.91 (−4.90)	−5.11 (−5.13)	−5.07 (−5.06)
18	0.05	1.50	−0.97 (−0.97)	1.15 (1.15)	−5.10 (−5.11)	−4.90 (−4.89)	−5.06 (−5.07)	−5.02 (−5.01)
20	0.05	1.49	−0.92 (−0.92)	1.17 (1.17)	−5.07 (−5.08)	−4.93 (−4.91)	−5.02 (−5.03)	−4.99 (−4.99)
40	0.03	1.50	−0.49 (−0.49)	1.42 (1.42)	−5.00 (−5.00)	−5.00 (−5.00)	−5.00 (−5.00)	−5.00 (−5.00)
60	0.00	1.51	0.00 (0.00)	1.51 (1.52)	−5.00 (−5.00)	−5.00 (−5.00)	−5.00 (−5.00)	−5.00 (−5.00)

the basis of a wide range of support types representative of both open and closed shells, a systematic sequence of implementation of these conditions has been shown. In this way, certain hitherto somewhat obscure points relating to the choice of appropriate kinematic boundary conditions for membrane and bending-disturbance displacements have been clarified.

Tables 1–3 clearly show that the agreement between the results of the analytical procedure presented in this work and those of a proven finite-element program is very close, the discrepancy being generally of the order of 1% (for the *main* values), and often much less. Thus the method may be used to obtain a reliable assessment of displacements in thin non-shallow spherical shells under axisymmetric loadings (e.g. domes, vessels and boiler-ends), especially when expensive numerical methods are to be avoided (in this respect, note the fineness of the mesh used to match the (main) analytical values). As expected, the computed displacements are generally small, though this may not always be the case for very thin shells with supports providing little or no lateral restraint. In these circumstances, the computation of displacements may constitute a necessary serviceability-limit check. This is clear by reference to the shell on horizontally-moving rollers, where both actions/stresses and displacements are much larger than for their encastré and fully-pinned counterparts, showing quite conclusively that the allowance of support 'breathing' in a shell is bad design practice unless, of course, such a 'breathing' is exactly perpendicular to the shell midsurface (Fig. 3(d)).

Summarizing, the two-stage method of analysis,

Table 3. Displacements and stresses for the illustrative example—pinned support on vertically-reacting rollers (Fig. 3(c))

ψ (deg.)	$v^T(\psi)$ (mm)	$w^T(\psi)$ (mm)	$\delta^T(\psi)$ (mm)	$y^T(\psi)$ (mm)	$\sigma_\varphi^T(\psi)$(N/mm^2) (*i*)	$\sigma_\varphi^T(\psi)$(N/mm^2) (*o*)	$\sigma_\theta^T(\psi)$(N/mm^2) (*i*)	$\sigma_\theta^T(\psi)$(N/mm^2) (*o*)
0	8.10	−14.04	16.21 (16.10)	0.00 (0.00)	−5.00 (−1.81)	−5.00 (−3.62)	+51.41 (+53.87)	+51.41 (+49.12)
1	8.39	−9.17	12.18 (12.18)	2.47 (2.40)	+12.90 (+14.55)	−22.90 (−23.04)	+42.36 (+44.84)	+35.20 (+33.02)
2	8.57	−4.70	8.53 (8.60)	4.78 (4.68)	+22.80 (+24.05)	−32.80 (−33.36)	+32.72 (+35.19)	+21.60 (+19.73)
3	8.68	−0.84	5.43 (5.53)	6.82 (6.70)	+26.78 (+27.78)	−36.78 (−37.70)	+23.49 (+25.78)	+10.78 (+9.19)
4	8.71	2.32	2.95 (3.05)	8.52 (8.40)	+26.69 (+27.52)	−36.69 (−37.87)	+15.29 (+17.26)	+2.62 (+1.29)
5	8.69	4.80	1.05 (1.15)	9.87 (9.75)	+24.03 (+24.73)	−34.03 (−35.34)	+8.41 (+10.00)	−3.20 (−4.28)
6	8.62	6.64	−0.30 (−0.24)	10.88 (10.78)	+19.95 (+20.55)	−29.95 (−31.29)	+2.95 (+4.13)	−7.03 (−7.92)
7	8.52	7.94	−1.21 (−1.17)	11.58 (11.51)	+15.29 (+15.77)	−25.29 (−26.58)	−1.18 (−0.39)	−9.30 (−10.00)
8	8.40	8.80	−1.76 (−1.75)	12.04 (11.99)	+10.69 (+11.05)	−20.69 (−21.82)	−4.11 (−3.66)	−10.38 (−10.94)
9	8.27	9.32	−2.04 (−2.05)	12.29 (12.26)	+6.45 (+6.72)	−16.45 (−17.42)	−6.05 (−5.89)	−10.63 (−11.05)
10	8.12	9.59	−2.12 (−2.16)	12.39 (12.37)	+2.84 (+2.98)	−12.84 (−13.59)	−7.19 (−7.25)	−10.32 (−10.63)
12	7.83	9.69	−1.96 (−2.01)	12.30 (12.30)	−2.38 (−2.41)	−7.62 (−7.97)	−7.85 (−8.18)	−8.90 (−9.04)
14	7.54	9.55	−1.63 (−1.68)	12.05 (12.05)	−5.17 (−5.31)	−4.83 (−4.87)	−7.37 (−7.76)	−7.30 (−7.33)
16	7.25	9.41	−1.32 (−1.35)	11.81 (11.79)	−6.25 (−6.43)	−3.75 (−3.61)	−6.55 (−6.89)	−6.05 (−6.02)
18	6.98	9.37	−1.09 (−1.11)	11.63 (11.59)	−6.36 (−6.53)	−3.64 (−3.44)	−5.82 (−6.04)	−5.28 (−5.21)
20	6.70	9.45	−0.94 (−0.95)	11.54 (11.49)	−6.04 (−6.18)	−3.96 (−3.76)	−5.31 (−5.42)	−4.89 (−4.81)
40	3.57	11.23	−0.49 (−0.49)	11.78 (11.72)	−5.00 (−5.00)	−5.00 (−5.01)	−5.01 (−5.01)	−5.01 (−5.01)
60	0.00	11.86	0.00 (0.00)	11.86 (11.81)	−5.00 (−5.00)	−5.00 (−5.00)	−5.00 (−5.00)	−5.00 (−5.00)

so widely used in the context of stress calculations, is also applicable for the determination of displacements, as has been demonstrated in the present article. Besides its practical relevance, such an approach has clear pedagogical advantages in the context of teaching of classical shell methods in general and of the Geckeler method in particular.

REFERENCES

1. J. W. Geckeler, Über die Festigkeit achsensymmetrischer Schalen, *Forschungsarb. Ingwes.*, **276**, 1–52 (1926).
2. S. P. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, McGraw-Hill, New York (1959).
3. W. Flügge, *Stresses in Shells*, Springer-Verlag, Berlin (1973).
4. H. Kraus, *Thin Elastic Shells*, John Wiley, New York (1967).
5. V. V. Novozhilov, *Thin Shell Theory*, Noordhoff, Groningen (1964).
6. M. N. Pavlović and A. Papamakarios, The structural behaviour of liquid-filled spherical vessels—Part I: Background and theoretical preliminaries, *Thin-Walled Struct.*, **10**, 195–214 (1990).
7. A. Zingoni and M. N. Pavlović, Effect of support conditions in liquid-filled spherical vessels—Part II: Inclined supports, *Proc. Instn Civil Engrs* (Part 2), **91**, 347–363 (1991).