Int. J. Appl. Engng Ed. Vol. 7, No. 1, pp. 64–67, 1991
Printed in Great Britain.

0742–0269/91 $3.00+0.00
Tempus Publications.

An Iterative Method in One-dimensional Compressible Flows*

R. Y. CHEN†
R. J. CHEN‡
New Jersey Institute of Technology, Newark, NJ 07102, *U.S.A.*

Nonlinear equations in one-dimensional compressible flow are commonly solved by the Newton-Raphson method. However, this method is potentially unstable. An iterative method is presented with guaranteed convergence properties.

INTRODUCTION

ONE-DIMENSIONAL compressible flow is an elective course for mechanical engineering seniors. The course covers isentropic flow, normal shocks, oblique shocks, convergent-divergent nozzle flow, flow with friction, flow with heat transfer, expansion waves, and two-dimensional method of characteristics. Solutions to these flows can be computed with a microcomputer.

In a compressible flow, properties are expressed as a nonlinear algebraic function of the Mach number. It is often necessary to solve an equation for the Mach number. A popular method of solution is the Newton-Raphson method. Unfortunately this method may fail to converge if the slope of the function is very small near the root. (Indeed, some programs merely throw up their hands in despair when Newton-Raphson iteration fails and return an incorrect answer with no warning message.) The purpose of this paper is to suggest an iterative method for finding the Mach number in a one-dimensional flow.

ITERATIVE METHOD

Let's consider an iterative method for some of the equations in one-dimensional compressible flow. In isentropic flow we have

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/(2\gamma-2)} \tag{1a}$$

We know ahead of time whether the solution is subsonic ($M < 1$) or supersonic ($M > 1$) by considering the physical conditions of the problem. If the solution is subsonic, we may rearrange the equation to obtain

$$M = \frac{A^*}{A}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/(2\gamma-2)} \tag{1b}$$

Taking this equation as the basis for an iterative method, we define

$$M_{i+1} = \frac{A^*}{A}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_i^2\right)\right]^{(\gamma+1)/(2\gamma-2)} \tag{1c}$$

For a given specific heat ratio γ, area ratio A/A^* = (local cross-sectional area)/(critical cross-sectional area), and initial value $0 < M_0 < 1$, the values M_i converge to a solution of the original equation: The right-hand side is a contraction; in particular, its derivative is less than $A^*/A < 1$ for $0 < M_i < 1$.

Conversely, if a supersonic solution to Equation (1a) is desired, we can solve for the right-hand instance of M^2, taking that as the basis for an iterative solution, starting with $M_0 > 1$. (Iteration converges by the Lemma in the Appendix, using $[1,\infty]$ and the facts that $A/A^* > 1$ and $1 < \gamma < 2$.)

Example 1: Isentropic flow with $A/A^* = 2.0$ and $\gamma = 1.40$.

Subsonic solution: $M = 0.30590$.

iteration	initial	1	2	3	4	5	6
M	1.0000	.50000	.33496	.30927	.30628	.30595	.30591
M	0.0000	.28935	.30413	.30571	.30588	.30590	.30590

Supersonic solution: $M = 2.1972$.

iteration	initial	1	2	3	4	5	9
M	1.0000	1.5999	1.9600	2.1120	2.1677	2.1871	2.1971
M	5.0000	2.8154	2.3821	2.2574	2.2174	2.2040	2.1973

For an adiabatic frictionalflow (Fanno line flow), we have

$$\frac{4fL_{max}}{D} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma}\ ln\left[\frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}\right] \tag{2}$$

where f (the average Fanning friction factor), L_{max} (the length of duct required to change the Mach

* Accepted 30 May 1990.
† Rong Y. Chen is Professor.
‡ Raymond Chen is a graduate student at the University of California at Berkeley.

number of the flow from M to 1) and D (the hydraulic diameter) are given, and a value of M is sought.

It was found that solving for the M^2 in the first term gives a subsonic iterative equation and solving for the M^2 term in the natural logarithm gives a supersonic iterative equation. In case of subsonic solution the initial guess of M must be greater than zero due to the logarithmic term.

Example 2: Fanno line flow with $4fL/D = 0.40$ and $\gamma = 1.40$.

Subsonic solution: $M = 0.62511$.

iteration	initial	1	2	3	5	9	14
M	1.0000	.80064	.70367	.65967	.63169	.62535	.62511
M	0.0001	.20652	.44121	.54629	.61039	.62459	.62510

Supersonic solution: $M = 2.3574$.

iteration	initial	1	2	3	5	7	10
M	1.0000	1.3453	1.7726	2.1018	2.3267	2.3542	2.3573
M	5.0000	2.9071	2.4475	2.3847	2.3601	2.3576	2.3574

For an isothermal friction flow:

$$\frac{4fL^*}{D} = \frac{1-\gamma M^2}{\gamma M^2} + ln\gamma M^2 \qquad (3)$$

where L^* is the length of duct required to change the Mach number of the flow from M to the critical Mach number. The critical Mach number is $M_c = 1/\sqrt{\gamma}$ or $\gamma M^2 = 1$ and $M_c = 0.845154$. If $\gamma M^2 < 1$, we may solve for γM^2 in the first term, and if $\gamma M^2 > 1$ we may solve for γM^2 in the logarithmic term.

Example 3: Isothermal frictional flow with $4fL/D = 0.40$ and $\gamma = 1.40$.

Subsonic solution: $M = 0.57257$.

iteration	initial	1	2	3	5	9	14
M	.84515	.71428	.64136	.60494	.57950	.57288	.57258
M	.00010	.19147	.40431	.49847	.55746	.57191	.57256

Supersonic solution: $M = 1.4288$.

iteration	initial	1	2	3	5	7	10
M	.84516	1.0323	1.2173	1.3347	1.4162	1.4272	1.4287
M	5.0000	1.6778	1.4991	1.4519	1.4315	1.4291	1.4288

For heat transfer flow (Rayleigh line flow):

$$\frac{p_0}{p_0^*} = \frac{\gamma+1}{1+\gamma M^2}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{\gamma/(\gamma-1)} \qquad (4)$$

where p_0 is the local stagnation pressure and p_0^* is the critical stagnation pressure for the flow at $M = 1$. This is similar to Equation (1a) and can be solved for the M^2 in the first factor if $M < 1$ and for the M^2 term in the second factor if $M > 1$.

Example 4: Rayleigh line flow with $Po/Po^* = 1.20$ and $\gamma = 1.40$.

Subsonic solution: $M = 0.29636$.

iteration	initial	1	2	3	5	9	14
M	1.0000	.84515	.70002	.57668	.41201	.30942	.29704
M	0.0001	.20100	.24892	.27138	.28902	.29569	.29632

Supersonic solution: $M = 1.6397$.

iteration	initial	1	2	3	5	9	14
M	1.0000	1.1493	1.2800	1.3853	1.5219	1.6178	1.6391
M	5.0000	2.9500	2.2986	2.0100	1.7750	1.6616	1.6422

For the Prandtl-Meyer flow we have

$$\nu = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{(\gamma-1)(M^2-1)}{(\gamma+1)}} + \tan^{-1}\sqrt{M^2-1} \qquad (5)$$

where ν is the Prandtle-Meyer angle. A value for M can be obtained by solving for M^2 in the first term.

Example 5: Prandtl-Meyer flow with $\nu = 11$ degrees and $\gamma = 1.40$.

Solution: $M = 1.4692$.

iteration	initial	1	2	3	5	9	16
M	1.0000	1.0183	1.0716	1.1508	1.3158	1.4508	1.4689
M	5.0000	2.0714	1.7066	1.5821	1.5000	1.4720	1.4692

Convergence for examples 2 through 5 can be accelerated by the following method: Start with $M_i = M_c$ for flow of $M > M_c$ (or $M_i) = 0.0$ for flow of $M < M_c$) and magnify the size of each step by a factor of two or $M_{i+1} = M_i + 2(f(M_i) - M_i)$.

Example 6: Repeat example 5 with step size magnified by a factor of two.

Solution: $M = 1.4692$.

iteration	initial	1	2	3	4	5	8
M	1.0000	1.0183	1.1025	1.2550	1.3917	1.4537	1.4692

One of the most complicated problems in one-dimensional compressible flow is predicting the location of a normal shock in a diverging section of a converging-diverging nozzle. When the specific heat ratio γ, the ratio of the nozzle exit area to throat area A_e/A_t, and the ratio of the back pressure to the inlet stagnation pressure p_b/p_{01}, are given, the flow is analyzed by assuming an isentropic flow throughout the nozzle except across the normal shock wave. To calculate the location of the normal shock (or the cross-sectional area at which the shock occurs), we may calculate the upstream Mach number before the shock M_1 and then use Equation (1a) to calculate the cross-sectional area of the nozzle at which the shock occurs. Following Olfe [1], we may express p_b/p_{01} as

$$\frac{p_b}{p_{01}} = \frac{p_e}{p_{01}} = \frac{p_e}{p_{02}}\frac{p_{02}}{p_{01}} = \frac{p_e}{p_{02}}\frac{A_1^*}{A_2^*} = \frac{p_e}{p_{02}}\frac{A_t}{A_e}\frac{A_e}{A_2^*} \qquad (6)$$

where p_e is the exit static pressure which is equal to the back pressure for subsonic flow. At the nozzle exit, p_{02} is the downstream stagnation pressure and A_1^* and A_2^* are respectively the upstream and downstream critical cross-sectional area. Using the isentropic relation [1,2]

$$\frac{p_e}{p_{02}} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\gamma/(1-\gamma)} \tag{7}$$

and A_e/A_2^* by Equation (1a) we can solve M^2 directly using the quadratic formula. From the calculated p_e/p_{02}, the stagnation pressure ratio across the shock wave p_{02}/p_{01} can be calculated. The stagnation pressure ratio is expressed in terms of the upstream Mach number M_1 [1,2] by

$$\frac{p_{02}}{p_{01}} = \frac{A_1^*}{A_2^*} = \left[\frac{(\gamma+1)M_1^2}{2 + (\gamma-1)M_1^2}\right]^{\gamma/(\gamma-1)} \left[\frac{\gamma+1}{2\gamma M_1^2 - (\gamma - 1)}\right]^{1/(\gamma-1)} \tag{8}$$

To determine the cross-sectional area A_1 at which the normal shock occurs we must solve Equation (8) for M_1 and use Equation (1a) to solve for A_1.

To use the Newton-Raphson iterative method, we manipulate Equation (8) into the form $f(M) = 0$, and make an initial guess for M_0. We use this guess as the basis for the recurrence

$$M_{i+1} = M_i - \frac{f(M_i)}{f'(M_i)} \tag{9}$$

which is applied repeatedly to obtain the root. This method gives very rapid convergence to the root when $f'(M_i)$ is very large, as shown in Fig. 1 for $p_b/p_{01} = 0.9$, $A_e/A_t = 2$ and $\gamma = 1.4$ (i.e. $p_{02}/p_{01} = 0.9655$) with an initial guess of $M_0 = 2$. However this method is very unstable and may fail to converge when $f'(M_i)$ is nearly equal to zero, for example, when $1 < M_i < 1.2$ in Fig. 1. In Fig. 1, we also see that for $M_i > 2.5$ the value of the iterate M_{i+1} may become negative. It was also found that for $1 < M_i < 1.09$ and $3.17 < M_i < 3.32$ the iterate

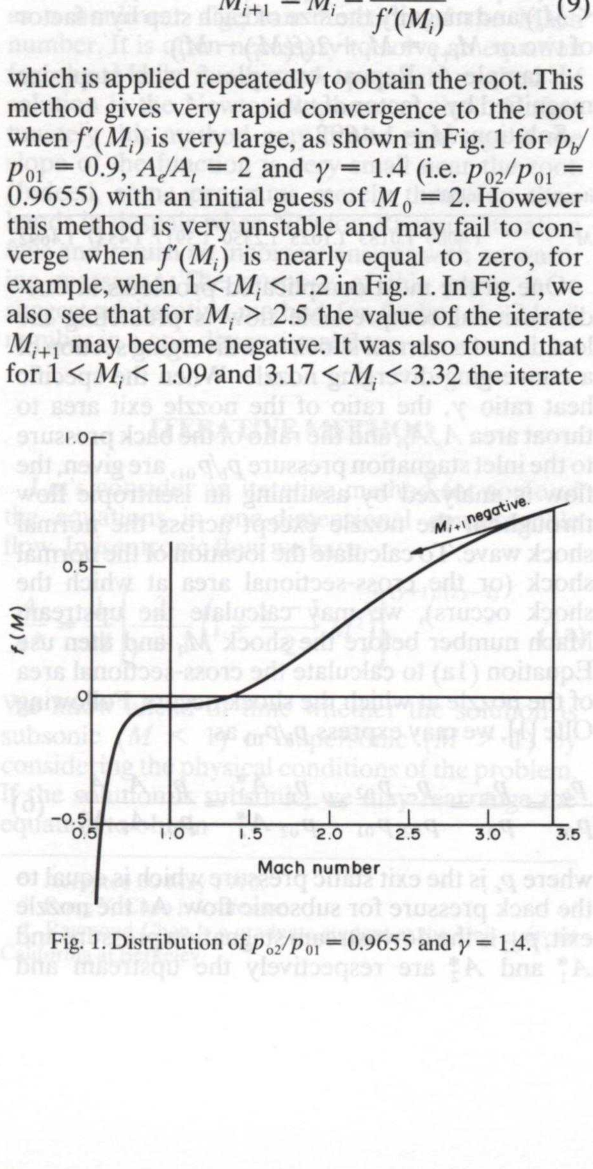

Fig. 1. Distribution of $p_{o2}/p_{01} = 0.9655$ and $\gamma = 1.4$.

M_{i+1} becomes so small that the second factor in Equation (8) becomes negative. This is particularly unfortunate, since an initial guess of $1 < M_0 < 1.09$ is not unreasonable.

A solution to Equation (8) can be obtained by a simple iterative method: Letting $N = M_1^2$ and solving for the M in the second factor, we obtain the recurrence

$$N_{i+1} = f(N_i) = \left[\frac{p_{01}}{p_{02}}\right]^{\gamma-1} \frac{(\gamma+1)^{\gamma+1}}{2\gamma(2/N_i) + \gamma - 1)^{\gamma}} + \frac{\gamma - 1}{2\gamma} \tag{10}$$

When we find a fixed point of the recurrence, its square root is the desired solution. We observe that f satisfies the conditions of the Lemma, and therefore iteration will converge. (With $[0,\infty]$ and the fact that $1 < \gamma < 2$.)

The rate of convergence depends on the particular value of γ, which determines the shape of the curve f: As $\gamma \rightarrow 1$, the rate of convergence slows. For commonly-used values of γ, however, the rate of convergence has been found to be reasonable.

Example 7: Normal shock in a convergent-divergent nozzle with $p_b/p_{01} = 0.90$, A_e/A_t and $\gamma = 1.40$ (i.e. $p_{02}/p_{01} = 0.96546$).

Solution: Upstream Mach number $M_1 = 1.3695$.

iteration	initial	1	2	3	5	21	100
M	1.0000	1.0061	1.0121	1.0183	1.0307	1.3527	1.3651
M	5.0000	2.9012	2.3881	2.1370	1.8791	1.4643	1.3701

Convergence can be acclerated by the following method: Start with $M_i = 1.01$ and magnify the size of each step by a factor of ten until the direction of the step changes, after which the usual iteration method is used. Put more explicitly, we let $M_{i+1} = M_i + 10(f(M_i) - M_i)$ until $f(M_i) < M_i$; after which we justs use $M_{i+1} = M_i + 2(f(M_i) - M_i)$.

Example 8: Do example 7 with step size magnified by 10.

Solution: Upstream Mach number $M_1 = 1.3695$.

iteration	initial	1	2	3	5	10	15
M	1.0100	1.1613	1.0778	1.1433	1.2676	1.3672	1.3695

APPENDIX

LEMMA. *Let $f(M)$ be a continuous function defined on $[a,b]$ (with a,b possibly infinite) satisfying the following conditions.*

1. $f'(M) < 0 < f'(M)$ *for* $a \leqslant M \leqslant b$
2. $f(a) > a$ *and* $f(b) < b$ *(if* $b = \infty$*, then we need* $f(M) < M$ *for* M *sufficiently large; similarly if* $a = -\infty$*),*

Then iteration will yield a fixed point.

Proof. It is immediate that a fixed point M^* exists and is unique. Observe that if $M^* < c$, then $M^* < f(c) < c$, so if our initial guess is too high, then $\langle M_i \rangle$ is a strictly decreasing sequence bounded from below (by M^*) and hence has a limit. But the limit is a fixed point since f is continuous. An analogous argument holds if our initial guess is too low.

REFERENCES

1. D. B. Olfe, *Fluid Mechanics Programs for the IBM PC*, McGraw-Hill (1987).
2. M. A. Saad, *Compressible Fluid Flow*, Prentice Hall (1985).

subject to the initial conditions

$$y(0) = 0,\ y^{(1)}(0) = 1,\ y^{(2)}(0) = -2,\ y^{(3)}(0) = 3.$$

The exact solution can be shown to be xe^{-x}. Using the notation in the paper this becomes

$$D_4y + D_3y + D_2y + ye^x(D_1y + y) = 1 + 2e^{-x}(x-2).$$

Consequently,

$$D_4y = 1 + 2e^{-x}(x-2) - D_3y - D_2y - y\,e^x(D_1y + y).$$

In terms of state variables this becomes

$$D_4y = 1 + 2e^{-x}(x\text{-}2) - y_1 - y_2 - y_4\,e^x(y_3 + y_4).$$

The right-hand side of this equation constructs the function f (see equation 7 and also line 120 of the program). Table 2 shows the computer results for two different increments. These results are directly compared to the exact values.

Table 2. Results for the example problem

x	y (h=.05)	y (h=.025)	y (Exact)
0	0	0	0
.05	.04535	.04643	.04756
.1	.08638	.08838	.09048
.15	.12339	.12619	.12911
.2	.15666	.16014	.16375
.25	.18648	.19025	.19470
.3	.21308	.21760	.22225
.35	.23670	.24162	.24664
.4	.25575	.26283	.26813
.45	.27590	.28144	.28693
.5	.29187	.29765	.30327
.55	.30568	.31167	.31732
.6	.31750	.32368	.32929
.65	.32750	.33386	.33933
.7	.33583	.34236	.34761
.75	.34266	.34936	.35427
.8	.34811	.35499	.35946
.85	.35233	.35941	.36330
.9	.35545	.36275	.36591
.95	.35760	.36513	.36740
1	.35889	.36669	.36788

CONCLUSIONS

Direct solution of higher-order differential equations can be very difficult and time consuming. Converting these equations to a system of first-order equations may be accomplished by introducing the state variables. However, this technique requires a workable computer program. These topics have been addressed in this paper. The proposed program is extremely compact and is easy to use. It offers the educator a powerful tool and can be integrated into a wide variety of areas such as dynamics and control.

REFERENCES

1. F. H. Raven, *Automatic Control Engineering*, 4th ed., McGraw-Hill (1987).
2. D. B. Miron, *Design of Feedback Control Systems*, Harcourt Brace Jovanovich (1989).
3. E. Mahajerin, A Unified computer program for the solution of ordinary differential equations, *Adv. Engng Software*, **9**(4), 222–224 (1987).