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Generalized One-dimensional Compressible Flow Techniques*

B. K. HODGE†

Mechanical and Nuclear Engineering Department, Mississippi State University, Mississippi State, MS 39762, U.S.A.

Techniques suitable for the analysis of a wide variety of generalized one-dimensional compressible flows are examined and assimilated. Examples of such flows involving situations with and without sonic points are analyzed, presented, and discussed. A BASIC computer program listing illustrating details of the implementation of the required techniques is provided in the Appendix. Usage of and justification for the generalized one-dimensional compressible flow analysis in an academic environment are discussed.

NOMENCLATURE

A Cross-sectional area
b Term defined by Eq. (14)
c Term defined by Eq. (14)
c_p Specific heat at constant pressure
D Diameter
f Fanning friction factor
G Defined in Eq. (1b)
k Runge-Kutta terms defined by Eq. (3b)
M Mach number
$\dot{m}$ Mass flow rate
P Pressure
s Entropy
T Temperature
x Spatial coordinate
γ Ratio of specific heats
Δx Integration step size
ρ Density
ψ Defined as $1 + (\gamma - 1)/2\, M^2$

Subscripts
b Back pressure
o Stagnation condition
o_i Stagnation condition at $x = 0$
1 Upstream station
2 Downstream station

INTRODUCTION

IN A previous article [1], Hodge explored the content and software for a microcomputer-based introductory compressible flow course. Two software elements, COMPQ and COMPINT, were introduced in that paper. COMPQ basically replaced the extensive tables and graphs usually associated with one-dimensional simple flows, and COMPINT represented an elementary implementation for generalized one-dimensional flows.

Simple flows are flows with only a single driving potential such as area change or friction or heat transfer or mass addition. In compressible flow the first three of these are usually called isentropic flow, Fanno flow, and Rayleigh flow, respectively. Generalized flows are flows with two or more of the driving potentials present. COMPINT is limited to flow situations that do not possess sonic points, a fairly stringent limitation.

The purpose of this paper is to present a more complete assimilation of generalized one-dimensional flow analysis techniques suitable for use in an instructional setting. Although Shapiro [2] first examined the generalized one-dimensional flow analysis technique, few compressible flow textbooks have fully exploited the procedure. Beans [3] in a technical paper and Zucrow and Hoffman [4] and Saad [5] do contain discussions of it, but particulars are relatively scant. In this paper, details of the techniques required are delineated and then illustrated using several examples. A listing of program COMPINT2, which can solve a wide variety of generalized compressible flow problems, is provided. The generalized procedure can be an attractive educational tool in the current personal computer computational environment.

THE GENERALIZED ONE-DIMENSIONAL COMPRESSIBLE FLOW TECHNIQUE

Many compressible flow textbooks, Zucrow and Hoffman [4] and Saad [5], for example, derive a system of differential equations describing the steady, one-dimensional flow of a calorically perfect gas in a duct with area change, friction, stagnation temperature change, and mass addition (or rejection) as the driving potentials. The differential equation with Mach number as the dependent variable and the spatial coordinate x as the independent variable is, from Saad

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† B. K. Hodge is Professor.

$$\frac{1}{M^2}\frac{\mathrm{d}M^2}{\mathrm{d}x}=\frac{\psi G(x,\gamma,M)}{1-M^2} \tag{1a}$$

where

$$\psi=1+\frac{\gamma-1}{2}M^2$$

$$G(x,\gamma,M)=-2\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x}+\gamma M^2\frac{4f}{D}+(1+\gamma M^2)\frac{1}{T_0}\frac{\mathrm{d}T_0}{\mathrm{d}x}+2(1+\gamma M^2)\frac{1}{\dot m}\frac{\mathrm{d}\dot m}{\mathrm{d}x} \tag{1b}$$

This expression is valid for mass injection perpendicular to the duct axis and with friction as the only drag force. If the variations of the four driving potentials with respect to x are known, then Eq. (1) can be integrated to yield the Mach number distribution as a function of x. The equation is well behaved and possesses no hidden problems except in the neighborhood of sonic point locations. Virtually any accurate numerical procedure can be used to integrate Equation (1). Because of its simplicity and accuracy, a fourth-order Runge-Kutta numerical procedure was used in this paper to integrate the differential equation. Given a differential equation of the functional form

$$\frac{\mathrm{d}M}{\mathrm{d}x}=f(x,\gamma,M) \tag{2}$$

with conditions at x known, the solution at location $x+\Delta x$ is

$$M(x+\Delta x)=M(x)+\frac{\Delta x}{6}(k_1+2k_2+2k_3+k_4) \tag{3a}$$

where

$$\begin{aligned}k_1&=f[x,\gamma,M(x)]\\k_2&=f[x+\Delta x/2,\gamma,M(x)+\Delta x k_1/2]\\k_3&=f[x+\Delta x/2,\gamma,M(x)+\Delta x k_2/2]\\k_4&=f[x+\Delta x,\gamma,M(x)+\Delta x k_3]\end{aligned} \tag{3b}$$

Additional details on Runge-Kutta procedures are provided in many numerical analysis textbooks; Conte and de Boor [6] is typical.

Normal shock waves are handled in one-dimensional generalized flow problems by continuing the integration procedure to the shock wave location. At the shock wave location, properties across the shock wave are calculated using the expressions that relate conditions across a normal shock wave. For completeness, the so-called 'jump conditions' are repeated here as

$$M_2^2=\frac{M_1^2+\dfrac{2}{\gamma-1}}{\dfrac{2\gamma}{\gamma-1}M_1^2-1}$$

$$\frac{P_2}{P_1}=\frac{2\gamma}{\gamma+1}M_1^2-\frac{\gamma-1}{\gamma+1}$$

$$\frac{\rho_2}{\rho_1}=\frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2} \tag{4}$$

$$\frac{T_2}{T_1}=\frac{1+\dfrac{\gamma-1}{2}M_1^2}{1+\dfrac{\gamma-1}{2}M_2^2}$$

where subscript 1 represents upstream and subscript 2 downstream conditions. After crossing the normal shock wave, the Runge-Kutta procedure is continued for the remainder of the x-range.

The physical properties are obtained from the integral relations, which can be derived [2,4,5] by considering various property ratios between two x-stations and the known driving potential variations. The integral relations are

$$\frac{T_2}{T_1}=\frac{T_{0_2}}{T_{0_1}}\frac{1+\dfrac{\gamma-1}{2}M_1^2}{1+\dfrac{\gamma-1}{2}M_2^2}$$

$$\frac{P_2}{P_1}=\frac{\dot m_2A_1M_1}{\dot m_1A_2M_2}\sqrt{T_2/T_1}$$

$$\frac{V_2}{V_1}=\frac{M_2}{M_1}\sqrt{T_2/T_1} \tag{5}$$

$$\frac{\rho_2}{\rho_1}=\frac{P_2}{P_1}\frac{T_1}{T_2}$$

$$\frac{P_{0_2}}{P_{0_1}}=\frac{P_2}{P_1}\left[\frac{T_{0_2}}{T_{0_1}}\frac{T_1}{T_2}\right]^{\gamma/(\gamma-1)}$$

$$\frac{\Delta s}{c_p}=ln\,T_2/T_1-\frac{\gamma-1}{\gamma}ln\,P_2/P_1$$

Zucrow and Hoffman [4] provide an outline for obtaining numerical solutions of generalized flows that possess no sonic point locations. A version of their four-step procedure is:

(1) Establish the intial conditions and develop models for the driving potentials present. These models must be functions of the independent variable x.
(2) Use the Runge-Kutta method to integrate the differential equation from x to $x+\Delta x$.
(3) Use the integral relations to obtain the physical properties at $x+\Delta x$.
(4) Repeat steps 2 and 3 over the x-range of interest.

Sonic points are of concern since a sonic point location is a singular point of the generalized differential equation relating Mach number and the driving potentials. The denominator of Eq. (1) becomes unbounded in the neighbourhood of $M = 1$, the sonic point. Thus, attempts to integrate through the sonic point using the Runge-Kutta procedure would result in 'overflows' and/or x-step size dependencies of the Mach numbers. Beans [3] and Zucrow and Hoffman [4] suggest procedures available to transit the sonic point location. We shall follow the technique proposed by Beans.

Equation (1) can be written as

$$(1-M^2)\frac{dM^2}{dx} = \psi\, G(x,\gamma,M) \tag{6}$$

Evaluation at the sonic point leads to

$$0 = G(x,\gamma,1) \tag{7}$$

since ψ is not zero at the sonic point. The sonic point is thus the root of Equation (7). For a non-simple flow, the sonic point need not be located at the minimum area. At the sonic point, Beans suggested that

$$\left.\frac{dM^2}{dx}\right|_{M=1} = \lim_{M\to 1} \psi \frac{G(x,\gamma,M)}{1-M^2} \tag{8}$$

is indeterminate since both the numerator and the denominator vanish. The value of dM^2/dx at the sonic point can then be evaluated by using l'Hospital's rule. Thus

$$\left.\frac{dM^2}{dx}\right|_{M=1} = \frac{\gamma+1}{2} \lim_{M\to 1} \frac{dG(x,\gamma,M)/dx}{-2M\, dM/dx} \tag{9}$$

since $M = M(x)$.

The term $dG(x,\gamma,M)/dx$ becomes

$$\frac{dG(x,\gamma,M)}{dx} = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial M}\frac{dM}{dx} \tag{10}$$

Evaluation of Eq. (10) at the sonic point yields

$$\frac{dG}{dx} = \left\{\frac{d}{dx}\left[-2\frac{1}{A}\frac{dA}{dx} + \gamma\frac{4f}{D} + (1+\gamma)\right]\frac{1}{T_0}\frac{dT_0}{dx} + 2(1+\gamma)\frac{1}{\dot m}\frac{d\dot m}{dx}\right\} + \frac{dM}{dx}\,2\gamma\left[\frac{4f}{D} + \frac{1}{T_0}\frac{dT_0}{dx} + 2\frac{1}{\dot m}\frac{d\dot m}{dx}\right] \tag{11}$$

Equation (9) can be stated as

$$\left[\frac{dM}{dx}\right]^2 = \frac{\gamma+1}{8}\frac{d}{dx}[G(x,\gamma,1)] - \frac{\gamma+1}{8}\left[\frac{dM}{dx}\right]2\gamma\left[\frac{4f}{D} + \frac{1}{T_0}\frac{dT_0}{dx} + 2\frac{1}{\dot m}\frac{d\dot m}{dx}\right] \tag{12}$$

This expression is a quadratic equation in terms of dM/dx at the sonic point. The solution for Eq. (12) is double valued; the negative value corresponding to decreasing Mach numbers, and the positive value corresponding to increasing Mach numbers. Hence

$$\frac{dM}{dx} = \frac{-b \pm \sqrt{b^2-4c}}{2} \tag{13}$$

where

$$b = \frac{\gamma+1}{8}\,2\gamma\left[\frac{4f}{D} + \frac{dT_0}{T_0\,dx} + \frac{2}{\dot m}\frac{d\dot m}{dx}\right]$$

$$c = \frac{\gamma+1}{8}\frac{d}{dx}[G(x,\gamma,1)] \tag{14}$$

The procedure of Beans [3] is simple: in the neighborhood of the sonic point, the limiting value of dM/dx from Eq. (13) is used instead of Equation (1). Singular behavior is thus avoided, and the solution proceeds smoothly through the sonic point. Since the value of dM/dx at the sonic point is double valued, the question of which root to use is pertinent. To achieve $M = 1$ at the sonic point, the positive root must be used when the flow is subsonic upstream of the sonic point. Supersonic flow downstream of the sonic point will be attained if the positive root, $dM/dx > 0$, is used downstream of the sonic point. If the negative root, $dM/dx < 0$, is used downstream of the sonic point, then the flow will be decelerated subsonically downstream of the sonic point.

The other remaining problem for non-simple duct flows possessing sonic points is that the entrance Mach number is unknown. This situation does not arise in simple area change flow since the sonic point is known (at the minimum area location) and since the area ratio determines the entrance Mach number. For the case of non-simple flow with a sonic point, the Mach number is one at the sonic point location, but the entrance Mach number is unknown. The Runge-Kutta method is used to integrate backwards, in the negative x-direction, from the sonic point to the duct entrance. The Mach number at the duct entrance can thus be established. Once the entrance Mach number is known, the Runge-Kutta method is integrated forward from the entrance, $x = 0$, and the properties evaluated using the integral relations. For both integrations, forwards and backwards, the appropriate limiting value of dM/dx from Eq. (13) is used in the neighborhood of the sonic point.

The procedure for applying the generalized one-dimensional flow analysis technique to ducts with sonic points then becomes:

1. Model the driving potentials as functions of x.
2. Locate the sonic point, x = XSP, using Eq. (7).
3. Evaluate $\mathrm{d}M/\mathrm{d}x$, DMDXMAX, at the sonic point using Eq. (13).
4. Starting at the sonic location (x = XSP), integrate backwards to establish the entrance Mach number, XMI. Use the positive root of Eq. (13) for this integration.
5. Starting at the entrance ($x = 0$), integrate forward to find the distributions of Mach number and properties. Upstream of the sonic point, the positive root of Eq. (13) is used: downstream the positive/negative is used for supersonic/subsonic flow.

GENERALIZED ONE-DIMENSIONAL FLOW EXAMPLES

The methodology is implemented in computer program COMPINT2; a listing is provided in Appendix A. The program is structured so that only the function definitions for the driving potentials and the initial conditions need to be altered for different problems. REM statements indicate the major steps in the program's logic. The procedures presented in the previous section can be better understood by considering examples.

Example 1

Air enters a duct with an initial Mach number of 2.0 and stagnation conditions of 1000 K and 200 kPa. The 2 m long duct is composed of a constant diameter section for 1 m followed by a 'sine-wave' shaped diameter distribution for 1 m. The initial diameter is 0.2 m, and the final diameter is 0.4 m. The friction factor is taken to be 0.005. The stagnation temperature varies in a linear fashion from 1000 K at the entrance to 600 K at the exit. The mass flow rate at the exit is 1.15 times the entrance mass flow rate. A normal shock wave is located 1.5 m from the entrance. The duct shape is sketched in Fig. 1(a). Determine the distributions of Mach number and static and stagnation pressure and temperature for this arrangement.

Solution

The problem possesses the driving potentials of area change, friction, heat rejection, and mass addition. Additionally, a shock wave location is specified and the duct shape involves two different diameter functions. With a supersonic entrance Mach number specified, a sonic point is not anticipated. Generalized problems with no sonic point locations and known inlet conditions can be easily solved using the procedure of Zucrow and Hoffman [4]. The driving potentials become:

Area Change

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = 0 \qquad 0 \leqslant x \leqslant \qquad (15)$$

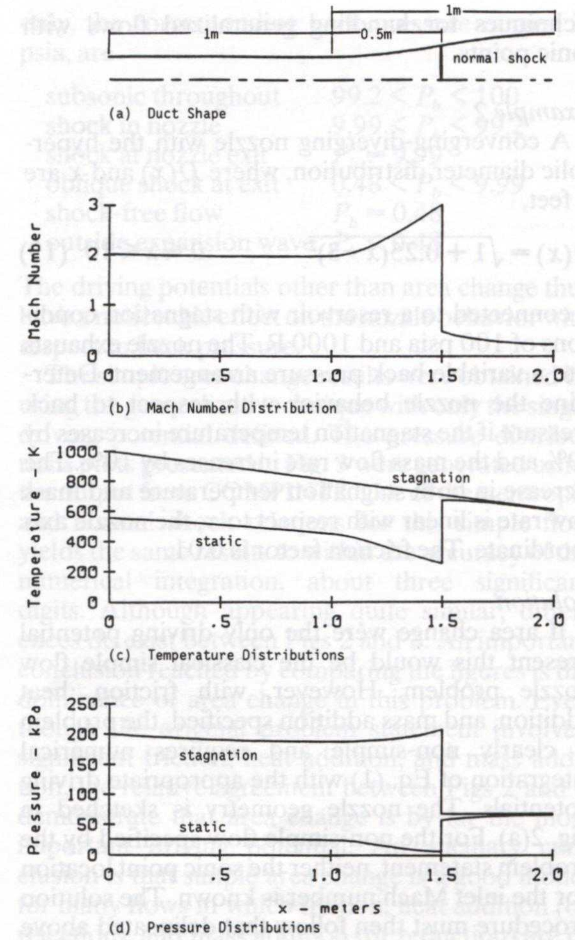

Fig. 1. Example Problem 1.

$$= \frac{0.2\pi \sin[\pi(x-1)]}{0.3-0.1\cos[\pi(x-1)]} \qquad 1 \leqslant x \leqslant 2$$

Heat Rejection

$$\frac{1}{T_0}\frac{\mathrm{d}T_0}{\mathrm{d}x} = \frac{-1}{5.0-x} \qquad 0 \leqslant x \leqslant 2 \qquad (16)$$

Friction

$$\frac{4f}{D} = \frac{4f}{0.2} \qquad 0 \leqslant x \leqslant 1$$

$$= \frac{4f}{0.3-0.1\cos[\pi(x-1)]} \qquad 1 \leqslant x \leqslant 2 \qquad (17)$$

Mass Addition

$$\frac{1}{\dot m}\frac{\mathrm{d}\dot m}{\mathrm{d}x} = \frac{0.075}{1+0.075\,x} \qquad 0 \leqslant x \leqslant 2. \qquad (18)$$

The procedure of Zucrow and Hoffman as implemented in COMPINT2 was used to obtain the solution for this problem. The resulting Mach number, temperature, and pressure distributions are shown in Figs 1(b), 1(c), and 1(d). This is a straightforward example since no sonic point is present.

The next example illustrates details of the techniques for handling generalized flows with sonic points.

Example 2

A converging-diverging nozzle with the hyperbolic diameter distribution, where $D(x)$ and x are in feet,

$$D(x)=\sqrt{1+0.25(x-3)^2} \qquad 0 \leqslant x \leqslant 10 \quad (19)$$

is connected to a reservoir with stagnation conditions of 100 psia and 1000 R. The nozzle exhausts into a variable back pressure arrangement. Determine the nozzle behavior with respect to back pressure if the stagnation temperature increases by 20% and the mass flow rate increases by 10%. The increase in both stagnation temperature and mass flow rate is linear with respect to x, the nozzle axis coordinate. The friction factor is 0.01.

Solution

If area change were the only driving potential present this would be the classical simple flow nozzle problem. However, with friction, heat addition, and mass addition specified, the problem is clearly non-simple and requires numerical integration of Eq. (1) with the appropriate driving potentials. The nozzle geometry is sketched in Fig. 2(a). For the non-simple flow specified by the problem statement, neither the sonic point location nor the inlet Mach number is known. The solution procedure must then follow that delineated above for geometries with a sonic point location. The program COMPINT2, as listed in Appendix A, was used to determine the back pressures corresponding to various positions of the normal shock wave location. The driving potentials are:

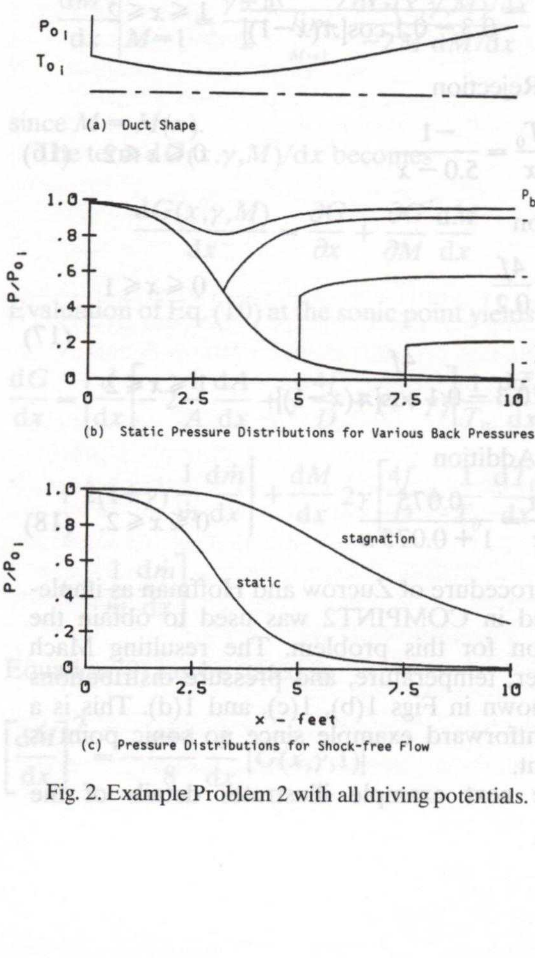

Fig. 2. Example Problem 2 with all driving potentials.

Area Change

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{x-3}{2[1+0.25(x-3)^2]} \quad (20)$$

Heat Addition

$$\frac{1}{T_0}\frac{\mathrm{d}T_0}{\mathrm{d}x}=\frac{1}{50+x} \quad (21)$$

Friction

$$\frac{4f}{D}=\frac{4f}{\sqrt{1+0.25(x-3)^2}} \quad (22)$$

Mass Addition

$$\frac{1}{\dot{m}}\frac{\mathrm{d}\dot{m}}{\mathrm{d}x}=\frac{1}{100+x} \quad (23)$$

The initial conditions for the program executions are

$$f=0.01 \qquad \Delta x=0.25 \qquad x\mathrm{Max}=10.0$$
$$x\mathrm{shock}=20,\,10,\,5,\,7.5$$
$$\frac{P_0}{P_{0_i}}=1 \qquad \frac{T_0}{T_{0_i}}=1 \quad (24)$$

Since the minimum area occurred at $x=3$, the initial guess for the sonic point location iteration was taken as $x=3$.

With the aforementioned coded into COMPINT2, the various back pressures required for different normal shock wave locations were found. The back pressure required for the shock to stand in the nozzle exit plane was generated by requiring a shock to be positioned at $x=10$, using the variable XSHOCK. A shock wave will be positioned at the nozzle exit for a back pressure of 9.34 psia. The program output for the normal shock to stand in the exit plane is provided in Table 1. In the output, the pressures and temperatures are nondimensionalized with the initial reservoir (stagnation) values of pressure and temperature. For the conditions of the problem, the back pressure required for shock-free flow is 1.14 psia.

The requirement for a normal shock wave of vanishing strength ($M_1=1$) to stand exactly at the sonic point, $x=3.148$, cannot be precisely attained because of the finite value of Δx used in the integration. The conditions are approximated by using the positive value of $\mathrm{d}M/\mathrm{d}x$ upstream of sonic point and the negative value of $\mathrm{d}M/\mathrm{d}x$ downstream of the sonic point. A normal shock wave standing at the sonic point corresponds to the maximum back pressure for the nozzle to be choked. For this problem, the maximum back pressure is 91.2 psia.

The behavior of the nozzle for the non-simple flow is thus established. The back pressures, expressed in psia, for the various ranges are

subsonic throughout	$91.2 < P_b < 100$
shock in nozzle	$9.34 < P_b < 91.2$

Table 1. COMPINT2 output from Example 2 with shock wave at nozzle exit.

```
EXAMPLE 2 - SHOCK AT NOZZLE EXIT

INPUT PARAMETERS:        GAMMA     F        DX
                         1.400     0.010    0.250
SONIC POINT INFORMATION:           XSP      DMDXMAX
                                   3.148    0.512
MACH NUMBER AT ENTRANCE ESTABLISHED TO BE 0.1636
```

I	X	MACH	PSTAG	PSTATIC	TSTAG	TSTATIC
1	0.0000	0.16356	1.00000	0.98150	1.00000	0.99468
2	0.2500	0.18574	0.99967	0.97590	1.00500	0.99811
3	0.5000	0.21197	0.99922	0.96842	1.01000	1.00100
4	0.7500	0.24312	0.99863	0.95839	1.01500	1.00314
5	1.0000	0.28019	0.99784	0.94488	1.02000	1.00423
6	1.2500	0.32436	0.99674	0.92669	1.02500	1.00388
7	1.5000	0.37693	0.99524	0.90228	1.03000	1.00154
8	1.7500	0.43926	0.99316	0.86989	1.03500	0.99654
9	2.0000	0.51264	0.99026	0.82773	1.04000	0.98807
10	2.2500	0.59802	0.98625	0.77442	1.04500	0.97524
11	2.5000	0.69577	0.98072	0.70970	1.05000	0.95731
12	2.7500	0.80535	0.97323	0.63507	1.05500	0.93386
13	3.0000	0.92514	0.96331	0.55410	1.06000	0.90507
14	3.2500	1.05259	0.95053	0.47176	1.06500	0.87181
15	3.5000	1.18425	0.93456	0.39336	1.07000	0.83562
16	3.7500	1.31683	0.91528	0.32284	1.07500	0.79818
17	4.0000	1.44711	0.89277	0.26241	1.08000	0.76119
18	4.2500	1.57257	0.86728	0.21248	1.08500	0.72595
19	4.5000	1.69139	0.83922	0.17224	1.09000	0.69331
20	4.7500	1.80245	0.80906	0.14028	1.09500	0.66373
21	5.0000	1.90515	0.77733	0.11509	1.10000	0.63734
22	5.2500	1.99935	0.74453	0.09525	1.10500	0.61407
23	5.5000	2.08517	0.71115	0.07959	1.11000	0.59371
24	5.7500	2.16292	0.67761	0.06716	1.11500	0.57604
25	6.0000	2.23302	0.64428	0.05722	1.12000	0.56076
26	6.2500	2.29597	0.61146	0.04921	1.12500	0.54763
27	6.5000	2.35225	0.57941	0.04270	1.13000	0.53641
28	6.7500	2.40237	0.54832	0.03737	1.13500	0.52686
29	7.0000	2.44684	0.51833	0.03296	1.14000	0.51879
30	7.2500	2.48611	0.48956	0.02928	1.14500	0.51204
31	7.5000	2.52063	0.46206	0.02619	1.15000	0.50645
32	7.7500	2.55081	0.43589	0.02357	1.15500	0.50188
33	8.0000	2.57704	0.41104	0.02135	1.16000	0.49823
34	8.2500	2.59966	0.38752	0.01943	1.16500	0.49540
35	8.5000	2.61901	0.36531	0.01778	1.17000	0.49329
36	8.7500	2.63538	0.34436	0.01634	1.17500	0.49183
37	9.0000	2.64903	0.32465	0.01508	1.18000	0.49096
38	9.2500	2.66023	0.30612	0.01398	1.18500	0.49061
39	9.5000	2.66919	0.28871	0.01300	1.19000	0.49074
40	9.7500	2.67613	0.27238	0.01214	1.19500	0.49130
41	10.0000	2.68123	0.25707	0.01136	1.20000	0.49225
	NORMAL SHOCK WAVE ENCOUNTERED					
41	10.0000	0.49712	0.11061	0.09342	1.20000	1.14348

shock at nozzle exit	$P_b = 9.34$
oblique shock at exit	$1.14 < P_b < 9.34$
shock-free flow	$P_b = 1.14$
outside expansion wave	$P_b < 1.14$

These are indicated in Fig. 2(b), which was generated from the outputs of the program executions. One of the most significant effects in this problem is the degradation of the stagnation pressure. Figure 2(c) shows the pressure distribution for supersonic shock-free nozzle flow for this problem. The loss of stagnation pressure is pronounced. An important point to note is that once the nozzle behavior is characterized and the generalized analysis available, the added complexity of multiple driving potentials can be readily handled.

For this same nozzle but with simple area change only, the corresponding back pressure ranges, in psia, are

subsonic throughout	$99.2 < P_b < 100$
shock in nozzle	$9.99 < P_b < 99.2$
shock at nozzle exit	$P_b = 9.99$
oblique shock at exit	$0.48 < P_b < 9.99$
shock-free flow	$P_b = 0.48$
outside expansion wave	$P_b < 0.48$

The driving potentials other than area change thus have a noticeable effect on the nozzle behavior with respect to back pressure.

The simple area change results were obtained by using the generalized technique with only the single driving potential defined. The pressure distributions plots presented in Fig. 3 were generated using the output from COMPINT2. Use of the isentropic and normal shock relations for this simple flow yields the same results to within the accuracy of the numerical integration, about three significant digits. Although appearing quite similar, differences do exist between Figs 2 and 3. An important conclusion reached by comparing the figures is the dominance of area change in this problem. Even though the original problem statement involved significant friction, heat addition, and mass addition, the relative agreement between Figs 2 and 3 demonstrate that area change is by far the most important driving potential. An ancillary conclusion is that simple area change is a good model for many flows in which friction, heat addition (or rejection), and mass addition (or reduction) are of secondary importance.

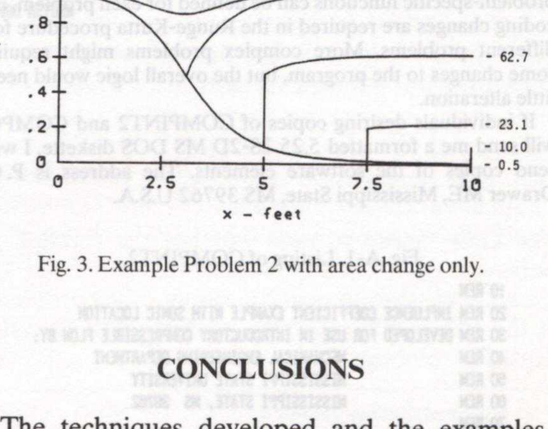

Fig. 3. Example Problem 2 with area change only.

CONCLUSIONS

The techniques developed and the examples illustrated herein provide an easily adapted extension to most compressible flow courses. The one-dimensional generalized analysis is straightforward to apply and permits the student to readily investigate non-simple flows. At Mississippi State University (MSU) the generalized flow concept has been used for the analysis of non-simple flows, as an educational 'what-if' tool to permit students to quantitatively see the importance of various effects (area change, friction, heat addition, and mass flow rate change), and as an adjunct to COMPQ which allows students to obtain the distributions of Mach number and properties for given simple flow situations.

When compressible flow is taught with the content and order as suggested by Hodge [1], little additional time is required to pursue the development of techniques for generalized one-dimensional compressible flow. At MSU an additional three (3) lecture periods (50 minutes each) are allocated to generalized flow and a major homework assignment, using COMPTIN2 as the basis, is required. Student response has been favorable for both the sequence of compressible flow coverage suggested by Hodge and the presentation of generalized flow techniques. Just the notion that they can handle more than one driving potential, as for the simple flows, seems to inspire confidence in the utility of one-dimensional gas dynamics.

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APPENDIX A

The listing of COMPINT 2 is presented in Fig. A-1. As the listing shows, the program is structured about function definitions. Three groups of function definitions are used: (1) the driving potentials for a specific problem, (2) the differential equation used for all problems, and (3) useful expressions used for all problems. The usual expressions relating static and stagnation temperatures and pressures are used in the order needed to compute the initial values of all properties. If an initial stagnation condition is taken as one, then the program output for that property is in the form of the ratio of the local property value divided by the initial stagnation property value at the entrance.

The Runge-Kutta procedure and the integral equations calculations are identified with REM statements. The differential equation function definition is used with the arguments of the differential equation function being other previously defined function definitions. Since the driving potentials and other problem-specific functions can be defined for each problem, no coding changes are required in the Runge-Kutta procedure for different problems. More complex problems might require some changes to the program, but the overall logic would need little alteration.

If individuals desiring copies of COMPINT2 and COMPQ will send me a formatted 5.25 2S-2D MS DOS diskette, I will send copies of the software elements. The address is P. O Drawer ME, Mississippi State, MS 39762 U.S.A.

Fig. A-1. Listing of COMPINT2.

```
10 REM
20 REM INFLUENCE COEFFICIENT EXAMPLE WITH SONIC LOCATION
30 REM DEVELOPED FOR USE IN INTRODUCTORY COMPRESSIBLE FLOW BY:
40 REM          MECHANICAL ENGINEERING DEPARTMENT
50 REM          MISSISSIPPI STATE UNIVERSITY
60 REM          MISSISSIPPI STATE, MS  39762
70 REM
80 PI = 3.141592€
90 REM
100 REM DRIVING POTENTIAL FUNCTION DEFINITIONS
110 REM
120 DEF FN AREADP(A) = (A - 3!)/(2!*(1! + .25*(A - 3!)^2))
130 DEF FN HEATDP(A) = 1!/(50! + A)
140 DEF FN DIA1(A) = SQR(1! + .25*(A - 3!)^2)
150 DEF FN MASSDP(A) = 1!/(100! + A) : DEF FN FRICDP(A,F) =
4!*F/SQR(1! + .25*(A - 3!)^2)
160 DEF FN TSTAG(A) = 1! + .02*A
170 DEF FN AREA(A) = .25*PI*(1! + .25*(A - 3!)^2)
180 DEF FN MDOT(A) = 1! + .01*A
190 DEF FN G(DPA,DPF,DPH,DPM,A,CM,GAM) = 2!*(-DPA + .5*GAM*CM*CM*DPF +
.5*(1! + GAM*CM*CM)*DPH + (1! + GAM*CM*CM)*DPM)
200 REM
210 REM DIFFERENTIAL EQUATION DEFINITION
220 REM
230 DEF FN FM(AET,AMT,DPA,DPF,DPH,DPM,A,CM,GAM) = CM*AET/AMT*(-DPA +
.5*GAM*CM*CM*DPF + .5*(1! + GAM*CM*CM)*DPH + (1! + GAM*CM*CM)*DPM)
240 REM
250 REM USEFUL FUNCTION DEFINITIONS
260 REM
270 DEF FN MT(CM) = 1! - CM*CM
280 DEF FN ET(CM) = 1! + .5*(GAM - 1!)*CM*CM
290 REM
300 REM INITIAL PARAMETERS
310 REM
320 F = .01
330 XMAX = 10! : XSHOCK = 10! : IS = 0
340 GAM = 1.4
350 PEX = GAM/(GAM - 1!)
360 DX = .25 : DXSAVE = DX
370 REM
380 REM BRANCH TO SUBROUTINES TO LOCATE SONIC POINT AND EVALUATE dM/dX
AT M = 1
390 REM
400 GOSUB 1250 : REM LOCATION OF SONIC POINT (XSP DETERMINED)
410 GOSUB 1000 : REM EVALUATION OF dM/dX AT M = 1
420 REM
430 REM BACKWARDS INTEGRATION TO ESTABLISH ENTRANCE MACH NUMBER
440 REM
450 X = XSP : DX = -DXSAVE : XM = 1! : I = 0
460 GOSUB 2000 : REM XMI DETERMINED
470 REM
480 REM INITIAL CONDITIONS OUTPUT
490 REM
500 XMI = XM : DX = DXSAVE : I = 1
```

```
510 TSI = FN TSTAG(0!)/FN ET(XMI)
520 PSTAGI = 1!: TSTAGI = FN TSTAG(0!)
530 PSI = PSTAGI/(FN ET(XMI)^PEX)
540 INPUT "ENTER THE TITLE";TT$
550 CLS
560 PRINT: PRINT TT$: PRINT
570 PRINT "INPUT PARAMETERS:      GAMMA      F           DX"
580 PRINT "                   "; USING "  #.###  ";GAM,F,DX
590 PRINT "SONIC POINT INFORMATION:          XSP        DMDXMAX"
600 PRINT "                              "; USING "   ##.###
";XSP,DMDXMAX
610 PRINT "MACH NUMBER AT ENTRANCE ESTABLISHED TO BE "; USING "
#.####";XMI
620 PRINT: PRINT "  I        X         MACH       PSTAG     PSTATIC
TSTAG     TSTATIC": PRINT
630 IS$ = STR$(I)
640 PRINT "  ";IS$;"  "; USING "  ###.###  ";X,XM;
650 PRINT USING "    #.##### ";PSTAGI,PSI,TSTAGI,TSI
660 REM END OF STATION OUTPUT
670 REM
680 REM FORWARD RUNGE-KUTTA INTEGRATION AND PROPERTY EVALUATION CALL
690 REM
700 X = 0! : DX = DXSAVE
710 GOSUB 2000
720 END
1000 REM
1010 REM EVALUATION OF DMDXMAX FOR SONIC POINT EVALUATION
1020 REM
1030 XSPP = XSP + .001 : XSPM = XSP - .001
1040 DGDX = (FN G(FN AREADP(XSPP),FN FRICDP(XSPP,F),FN HEATDP(XSPP),FN
MASSDP(XSPP),XSPP,1!,GAM) - FN G(FN AREADP(XSPM), FN FRICDP(XSPM,F),FN
HEATDP(XSPM),FN MASSDP(XSPM),XSPM,1!,GAM))/(XSPP - XSPM)
1050 CTERM = (GAM + 1!)*DGDX/8!
1060 BTERM = (GAM + 1!)*(2!*GAM/8!)*(FN FRICDP(XSP,F) + FN HEATDP(XSP)
+ 2!*FN MASSDP(XSP))
1070 DMDX1 = -.5*BTERM + .5*SQR(BTERM*BTERM - 4!*CTERM) : DMDX2 = -
.5*BTERM - SQR(BTERM*BTERM - 4!*CTERM)
1080 DMDXMAX = DMDX1 : REM DMDXMAX FOR SUPERSONIC FLOW CHOSEN
1090 RETURN
1250 REM
1260 REM NEWTON-RAPHSON TO LOCATE SONIC POINT
1270 REM
1280 XINT = 3!
1290 XPLUS = XINT + .001 : XMIN = XINT - .001
1300 DGDX = (FN G(FN AREADP(XPLUS),FN FRICDP(XPLUS,F),FN
HEATDP(XPLUS),FN MASSDP(XPLUS),XPLUS,1!,GAM) - FN G(FN AREADP(XMIN),FN
FRICDP(XMIN,F),FN HEATDP(XMIN),FN MASSDP(XMIN),XMIN,1!,GAM))/(XPLUS -
XMIN)
1310 XNEW = XINT - FN G(FN AREADP(XINT),FN FRICDP(XINT,F),FN
HEATDP(XINT),FN MASSDP(XINT),XINT,1!,GAM)/DGDX
1320 IF ABS(XNEW - XINT) < .0001 THEN GOTO 1340
1330 XINT = XNEW : GOTO 1290
1340 XSP = XINT
1350 RETURN
1500 REM
1510 REM SHOCK WAVE JUMP CONDITIONS
1520 REM
1530 IS = 1
1540 MO = XM
1550 XM = SQR((XM*XM + 2!/(GAM - 1!))/(2!*GAM/(GAM - 1!)*XM*XM - 1!))
1560 PS = PS*(2!*GAM/(GAM + 1!)*MO*MO - (GAM - 1!)/(GAM + 1!))
1570 TS = TS*FN ET(MO)/FN ET(XM)
1580 PSTAG = PS*FN ET(XM)^PEX
1590 DELS = LOG(TS/TSI) - (GAM - 1!)*LOG(PS/PSI)/GAM
1600 PRINT "              NORMAL SHOCK WAVE ENCOUNTERED"
1610 PRINT "          ";USING "  ###.###  ";X,XM;
1620 PRINT USING "    #.##### ";PSTAG,PS,FN TSTAG(X),TS
1630 RETURN
2000 REM
2010 REM RUNGE-KUTTA INTEGRATION ROUTINE
2020 REM
2030 I = I + 1 : MSONIC = .05
2040 REM FIRST RUNGE-KUTTA PASS
2050 IF ABS(XM - 1!) < MSONIC THEN GOTO 2070
2060 K1 = FN FM(FN ET(XM),FN MT(XM),FN AREADP(X),FN FRICDP(X,F),FN
HEATDP(X),FN MASSDP(X),X,XM,GAM) : GOTO 2090
2070 K1 = DMDXMAX : REM IF X > XSP THEN K1 = DMDX2
2080 REM SECOND RUNGE-KUTTA PASS
2090 XA = X + .5*DX
2100 XMA = XM + .5*DX*K1
2110 IF ABS(XMA - 1!) < MSONIC THEN GOTO 2130
2120 K2 = FN FM(FN ET(XMA),FN MT(XMA),FN AREADP(XA),FN FRICDP(XA,F),FN
HEATDP(XA),FN MASSDP(XA),XA,XMA,GAM) : GOTO 2150
2130 K2 = DMDXMAX : REM IF X > XSP THEN K2 = DMDX2
2140 REM THIRD RUNGE-KUTTA PASS
2150 XMA = XM + .5*DX*K2
2160 IF ABS(XMA - 1!) < MSONIC THEN GOTO 2180
2170 K3 = FN FM(FN ET(XMA),FN MT(XMA),FN AREADP(XA),FN FRICDP(XA,F),FN
HEATDP(XA),FN MASSDP(XA),XA,XMA,GAM) : GOTO 2200
2180 K3 = DMDXMAX : REM IF X > XSP THEN K3 = DMDX2
2190 REM FOURTH RUNGE-KUTTA PASS
2200 XA = X + DX
2210 XMA = XM + DX*K3
2220 IF ABS(XMA - 1!) < MSONIC THEN GOTO 2240
2230 K4 = FN FM(FN ET(XMA),FN MT(XMA),FN AREADP(XA),FN FRICDP(XA,F),FN
HEATDP(XA),FN MASSDP(XA),XA,XMA,GAM) : GOTO 2260
2240 K4 = DMDXMAX : REM IF X > XSP THEN K4 = DMDX2
2250 REM MACH NUMBER AT X + DX
2260 XM = XM + DX*(K1 + 2!*(K2 + K3) + K4)/6!
2270 X = XA
2280 IF DX < 0! THEN GOTO 2440
2290 REM
2300 REM PROPERTY CALCULATIONS
2310 REM
2320 T2OT1 = FN TSTAG(X)*FN ET(XMI)/(FN TSTAG(0!)*FN ET(XM))
2330 TS = T2OT1*TSI
2340 TSTAG = FN TSTAG(X)
2350 P2OP1 = FN MDOT(X)*FN AREA(0!)*XMI*SQR(T2OT1)/(FN MDOT(0!)*FN
AREA(X)*XM)
2360 PS = P2OP1*PSI
2370 PSTAG = P2OP1*PSTAGI*(FN ET(XM)/FN ET(XMI))^PEX
2380 IS$ = STR$(I)
2390 PRINT "  ";IS$;"  "; USING "  ###.###  ";X,XM;
2400 PRINT USING "    #.##### ";PSTAG,PS,TSTAG,TS
2410 REM CHECK FOR SHOCK WAVE LOCATION
2420 IF X > (XSHOCK - .5*DX) AND IS = 0 THEN GOSUB 1500
2430 REM CHECK ON END OF INTEGRATION SEQUENCES
2440 IF DX < 0! THEN GOTO 2470
2450 IF (X + .001) < XMAX THEN GOTO 2030
2460 RETURN
2470 IF ABS(DX) > X THEN DX = -X
2480 IF X > .0001 THEN GOTO 2030
2490 RETURN
```