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Process Path for Simple Mass Addition in One-dimensional Compressible Flow*

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The simple mass addition (or removal) equivalent to the Rayleigh or Fanno line is examined on a Mollier diagram. The equations for the simple mass addition process are reviewed, and the procedure required to generate the mass addition process path is examined. Salient behavioral features on the process path are discussed and an example given. The simple mass addition or removal process path on a Mollier diagram is an effective pedagogical device for presenting process limits and behavior in introductory compressible flow courses.

NOMENCLATURE

A	Cross-sectional area
c_p	Specific heat at constant pressure
h	Enthalpy
M	Mach number
$\dot{m}$	Mass flow rate
P	Pressure
s	Entropy
T	Temperature
V	Velocity
x	Spatial coordinate
γ	Ratio of specific heats
Δs	Defined as $s - s^*$
ρ	Density
Subscripts	
o	Stagnation condition
o_1	Stagnation condition at $x = 0$
o_2	Stagnation condition at process end
1	Upstream station
2	Downstream station
Superscripts	
*	Evaluated at the sonic point

INTRODUCTION

ONE-DIMENSIONAL compressible flows in which only a single driving potential is present are called simple flows [1,2,3]. Because Mach number and property relations for simple flows can be expressed in closed form, such flows are usually treated individually. Most introductory compressible flow courses and most compressible flow textbooks discuss simple area change, usually called isentropic flow, simple heat addition (or rejection), usually called Rayleigh flow, and simple frictional flow, usually called Fanno flow. Simple mass addition or removal, if it is referred to at all, generally receives only limited treatment [2,3]. For simplicity the term 'simple mass addition or removal' will be used hereafter as 'simple mass addition' with the removal option understood.

Rayleigh and Fanno flow coverage is usually centered about the representation of the process path on a Mollier (enthalpy-entropy) diagram. These representations are called Rayleigh or Fanno lines for constant area and mass velocity (ρV) compressible flows with heat addition or friction, respectively. The Rayleigh or Fanno lines are unique for a given mass velocity and given stagnation conditions. The usual pedagogical procedure is to construct a dimensionless form of the Rayleigh or Fanno line and to use that representation to illustrate salient features and characteristics, including limiting conditions, of the flow.

As far as the author can determine, the process path for simple mass addition has not been utilized or referenced in the literature. The purpose of this note is to examine and illustrate the pedagogical uses of the mass addition process path. The usual development for simple mass addition is reviewed, and the procedure for constructing the process path is delineated. The instructional utility of the mass addition process path on the Mollier diagram is discussed and an example presented. Such an approach to simple mass addition has proven effective in introductory compressible flow instruction.

Analysis of Simple Mass Addition

When the only driving potential for a steady, one-dimensional gas flow is mass flow rate change, then the mass addition process is defined. If the injectant gas is the same as the main stream gas, if the stagnation enthalpies of the injectant and the main stream gases are the same, and if the gas is considered calorically perfect, then simple mass

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addition is defined. Differential expressions for simple mass addition can be derived by applying the conservation equations to an elemental control volume such as in Fig. 1.

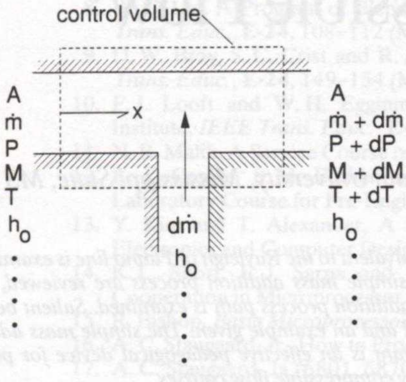

Fig. 1. Control volume for simple mass addition.

Conservation of mass becomes

$$\frac{\mathrm{d}\dot{m}}{\dot{m}} = \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}V}{V} \tag{1}$$

Since the stagnation enthalpies of both the main stream and the injected gas are the same, conservation of energy reduces to

$$\mathrm{d}h_o = 0. \tag{2}$$

Conservation of momentum, assuming the gas is injected with no x-component of velocity, becomes for the situation illustrated in Fig. 1

$$\mathrm{d}P + \rho V \mathrm{d}V + \rho V^2 \frac{\mathrm{d}\dot{m}}{\dot{m}} = 0 \tag{3}$$

If expressions for the equation of state, the speed of sound, the stagnation pressure, and the change in entropy are differentiated, then

$$\frac{\mathrm{d}P}{P} = \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}T}{T} \tag{4a}$$

$$\frac{\mathrm{d}M}{M} = \frac{\mathrm{d}V}{V} - \frac{1}{2}\frac{\mathrm{d}T}{T} \tag{4b}$$

$$\frac{\mathrm{d}P_0}{P_0} = \frac{\mathrm{d}P}{P} + \frac{\gamma M^2}{1+\frac{\gamma-1}{2}M^2}\frac{\mathrm{d}M}{M} \tag{4c}$$

$$\frac{\mathrm{d}s}{c_p} = -\frac{\gamma-1}{\gamma}\frac{\mathrm{d}P_0}{P_0} \tag{4d}$$

By algebraic reduction, Equations (1)–(4) can be cast in the forms of differential property relations ($\mathrm{d}P/P$, for example) as functions of Mach number and $\mathrm{d}\dot{m}/\dot{m}$. The results of the reduction process are given in Table 1. A positive value of $\mathrm{d}\dot{m}/\dot{m}$ signifies mass addition, and a negative value of $\mathrm{d}\dot{m}/\dot{m}$ signifies mass removal. The form of these

Table 1. Simple mass addition differential forms (normal injection)

$$\frac{\mathrm{d}M^2}{M^2} = \frac{2(1+\gamma M^2)\psi}{1-M^2}\frac{\mathrm{d}\dot{m}}{\dot{m}} \qquad \frac{\mathrm{d}\rho}{\rho} = -\frac{(\gamma+1)M^2}{1-M^2}\frac{\mathrm{d}\dot{m}}{\dot{m}}$$

$$\frac{\mathrm{d}V}{V} = \frac{1+\gamma M^2}{1-M^2}\frac{\mathrm{d}\dot{m}}{\dot{m}} \qquad \frac{\mathrm{d}P}{P} = -\frac{2\gamma M^2\psi}{1-M^2}\frac{\mathrm{d}\dot{m}}{\dot{m}}$$

$$\frac{\mathrm{d}T}{T} = -\frac{(\gamma-1)M^2(1+\gamma M^2)}{1-M^2}\frac{\mathrm{d}\dot{m}}{\dot{m}} \qquad \frac{\mathrm{d}P_0}{P_0} = -\gamma M^2\frac{\mathrm{d}\dot{m}}{\dot{m}}$$

$$\frac{\mathrm{d}s}{c_p} = (\gamma-1)M^2\frac{\mathrm{d}\dot{m}}{\dot{m}} \qquad \psi = 1 + \frac{\gamma+1}{2}M^2$$

expressions is called the influence coefficient form [1,2,3] since the differential change in a dependent variable is represented by the partial derivative of that variable with respect to the independent variable, the mass flow rate change. The expressions given in Table 1 could have also been obtained by using the table of influence coefficients developed for generalized one-dimensional flow; see Zucrow and Hoffman [2].

Integration of the entries in Table 1 with conditions referenced to the sonic point produces the entries of Table 2. As with the other simple flows, the starred variables denote conditions at the sonic location. The entropy change is referenced to the sonic state, so $\Delta s = s - s^*$. The expressions in Table 2 can be tabulated as functions of Mach number and specific heat ratio in much the same fashion as the isentropic, Rayleigh, or Fanno relations in the usual compressible flow textbook [2]. Shapiro [3] presents a plot of the relations from Table 2 for $\gamma = 1.4$. Hodge [1] in the personal computer utility program COMPQ provides an option for simple mass addition that alleviates table lookups. Any of these sources can be used to work simple mass addition problems in much the same fashion as the more familiar isentropic, Rayleigh, or Fanno flow problems.

Table 2. Simple mass addition closed form relations (normal injection)

$$\frac{\dot{m}}{\dot{m}^*} = \frac{M\sqrt{2(\gamma+1)\psi}}{1+\gamma M^2} \qquad \frac{\rho}{\rho^*} = \frac{2\psi}{1+\gamma M^2}$$

$$\frac{P}{P^*} = \frac{\gamma+1}{1+\gamma M^2} \qquad A = A^* = \text{const}$$

$$\frac{T}{T^*} = \left[\left(\frac{2}{\gamma+1}\right)\psi\right]^{-1} \qquad T_0 = T_0^* = \text{const}$$

$$\frac{V}{V^*} = M\left[\left(\frac{2}{\gamma+1}\right)\psi\right]^{-1/2} \qquad \psi = 1 + \frac{\gamma-1}{2}M^2$$

$$\frac{P_0}{P_0^*} = \frac{\gamma+1}{1+\gamma M^2}\left[\left(\frac{2}{\gamma+1}\right)\psi\right]^{\gamma/\gamma-1}$$

$$\frac{\Delta s}{c_p} = \ln\left\{\left[\left(\frac{2}{\gamma+1}\right)\psi\right]^{-1}\left(\frac{1+\gamma M^2}{\gamma+1}\right)^{(\gamma-1)/\gamma}\right\}$$

For simple mass addition the increase in mass flow rate is the important parameter; no distinction is made involving the distance over which the fluid injection occurs. The differential property relations, Table 1, for entropy and stagnation pressure show that mass addition produces an increase in entropy and a decrease in stagnation pressure; mass removal produces a decrease in entropy and an increase in the stagnation pressure. Figure 2 schematically represents the conditions and nomenclature for the simple mass addition process.

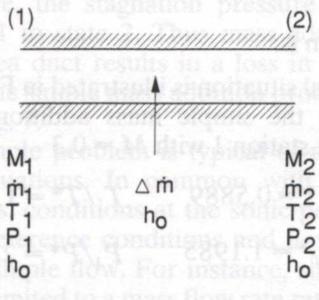

Fig. 2. Simple mass addition nomenclature.

Since the mass flow rate is not constant in this simple flow, the equivalent of the Rayleigh and Fanno lines, which are for constant mass flow rates, is slightly different. However, the process can be represented on a Mollier diagram. Such a representation is a convenient mechanism for exploring this simple flow. The Mollier diagram process path representation for simple mass addition represents the loci of states for that particular process. The change in entropy (referenced to the sonic state) is, from Table 2,

$$\frac{\Delta s}{c_p} = ln\left[\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{-1}\left(\frac{1+\gamma M^2}{\gamma+1}\right)^{\gamma-1/\gamma}\right] \tag{5}$$

The static temperature ratio, T/T^*, as a function of Mach number can be expressed as

$$\frac{T}{T^*} = \left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{-1} \tag{6}$$

The dimensionless simple mass addition process path can be generated by using Mach number as the parameter in conjunction with the above two expressions. For a calorically perfect gas the temperature ratio is equal to the enthalpy ratio since

$$\frac{T}{T^*} = \frac{c_p T}{c_p T^*} = \frac{h}{h^*} \tag{7}$$

The dimensionless plot on a Mollier diagram for a specific heat ratio of 1.4 is given in Fig. 3. Mach number as a function of $\Delta s/c_p$ for $\gamma = 1.4$ is plotted as Fig. 4.

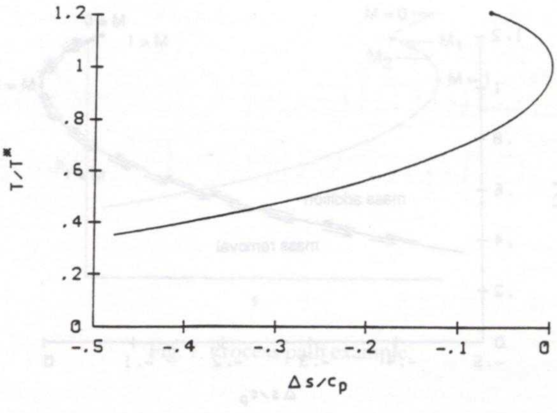

Fig. 3. Simple mass addition process path on a Mollier Diagram.

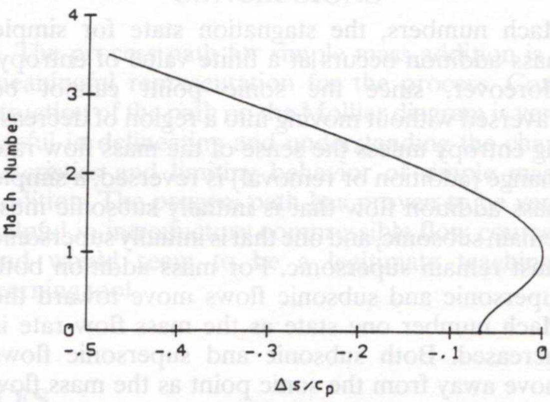

Fig. 4. Mach number versus dimensionless entropy for a simple mass addition process.

By observation, Fig. 4 suggests that the maximum entropy state is the sonic state. This can be verified analytically by finding the Mach number corresponding to the maximum entropy. The change in entropy for a simple mass addition flow, cast in terms of the Mach number, is given in Equation (5). To find the Mach number at the maximum entropy state, the derivative of $\Delta s/c_p$ or $\exp(\Delta s/c_p)$ with respect to the Mach number is taken and set equal to zero. The result is

$$M = 1. \tag{8}$$

For a calorically perfect gas undergoing a simple mass addition, the state of maximum entropy is thus the sonic state, $M = 1$.

Information about the mass addition process is schematically shown on Fig. 5. Movement is to the right (in the direction of increasing entropy) on a Mollier diagram for mass addition and to the left (in the region of decreasing entropy) for mass removal. The sonic state occurs at the point of maximum entropy. Subsonic Mach numbers are associated with the large temperature or enthalpy ratios, and supersonic Mach numbers with the smaller values of the ratios. Thus, the upper portion of the process line corresponds to subsonic flow and the lower portion to supersonic flow. Unlike the Rayleigh or Fanno lines that possess unbounded (negative) values of entropy for low

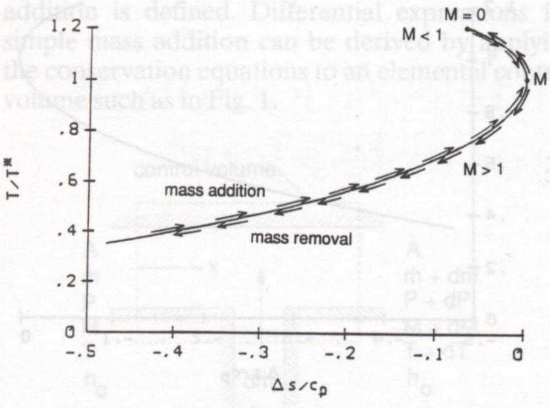

Fig. 5. Simple mass addition process path features.

Mach numbers, the stagnation state for simple mass addition occurs at a finite value of entropy. Moreover, since the sonic point cannot be traversed without moving into a region of decreasing entropy unless the sense of the mass flow rate change (addition or removal) is reversed, a simple mass addition flow that is initially subsonic must remain subsonic, and one that is initially supersonic must remain supersonic. For mass addition both supersonic and subsonic flows move toward the Mach number one state as the mass flow rate is increased. Both subsonic and supersonic flows move away from the sonic point as the mass flow rate is reduced. The meaning of $\dot{m}^*$ is now clear; it is the mass flow rate required for the Mach number to become one. As such it represents the limit of mass addition for subsonic or supersonic flow. Conditions at the location of Mach one in a simple mass addition flow, as with the other simple flows, are unique for that given flow and can be used as reference conditions to aid in calculations.

Just like Rayleigh flow, simple mass addition provides, at least in theory, an alternative to the converging-diverging nozzle for achieving supersonic flow. If the mass flow rate is increased in a subsonic flow from a reservoir in a simple mass addition situation, Mach one can be obtained. If after Mach one is attained, the mass flow rate is reduced, the flow then expands supersonically. Thus with exactly the right amount of mass addition followed by mass removal, supersonic flow in a constant area duct can be achieved. Because of effects, such as wall friction and lack of flow uniformity, not considered in simple mass addition, the sequence is not really feasible, and to the best of the author's knowledge, no such device has ever been successful in obtaining steady supersonic flow.

Duct length does not enter into any of the simple mass addition flow expressions given in Table 2. For a given process, the mass addition (or removal) and the initial state determine the final state. Friction, which depends on the length, is neglected in simple mass addition flows. Duct length can enter into the property distributions, P versus x for example, for a simple mass addition flow in which mass is added or removed at some specified rate in the one-dimensional duct.

Consider the following example:

'Air with initial stagnation conditions of 600 K and 1 MPa flows at a Mach number of 0.3 at the entrance to a constant area porous-walled duct. During passage through the duct, the mass flow rate is increased by 50%. If the stagnation enthalpy of the injected gas is the same as for the mainstream, find the exit conditions and sketch the process on the simple mass addition Mollier diagram.'

The solution is:

The physical situation is illustrated in Fig. 6. Using Table 2 or the simple mass addition tables or COMPQ at station 1 with $M = 0.3$

$$\dot{m}_1/\dot{m}^* = 0.5889 \qquad T_1/T^* = 1.1788$$
$$P_{01}/P_0^* = 1.1985 \qquad P_1/P^* = 2.1314. \tag{9}$$

After the mass addition, $\dot{m}_2 = 1.5\,\dot{m}_1$, so that

$$\dot{m}_2/\dot{m}^* = (\dot{m}_2/\dot{m}_1)(\dot{m}_1/\dot{m}^*)$$
$$= (1.5)(0.5889) = 0.88335. \tag{10}$$

with the value of $\dot{m}_2/\dot{m}^*$known, the value of M_2 can be found. Since $\dot{m}/\dot{m}^*$ is double valued in Mach number, both supersonic and subsonic flow are possible. Earlier we had established that with the same sense (addition or removal) of mass flow rate change, a process initially subsonic must remain subsonic and a process supersonic must remain supersonic. The process under consideration in this problem started subsonically, so that subsonic flow must also exist at station 2. After the mass addition the Mach number is 0.5663. The static and stagnation properties at station 2 can be computed from the various property ratios

$$T_2/T_1 = (T_2/T^*)/(T_1/T^*)$$
$$= 1.1277/1.1788 = 0.9567$$
$$P_{02}/P_{01} = (P_{02}/P_0^*)/(P_{01}/P_0^*)$$
$$= 1.0877/1.1985 = 0.9076 \tag{11}$$
$$P_2/P_1 = (P_2/P^*)/(P_1/P^*)$$
$$= 1.6563/2.1314 = 0.7771.$$

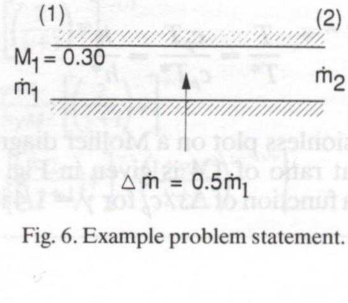

Fig. 6. Example problem statement.

At the initial Mach number of 0.3, the static properties are 939.5 kPa and 589.4 K. The properties at the end of the mass addition process (station 2) then become

$$\begin{aligned} M_2 &= 0.5663 \\ T_{02} &= T_{01} = 600\ \text{K} \\ T_2 &= (0.9567)\,589.4 = 563.9\ \text{K} \\ P_0 &= (0.9076)\,1. = 907.6\ \text{kPa} \\ P_2 &= (0.7771)\,939.5 = 730.1\ \text{kPa} \end{aligned} \tag{12}$$

Because the change in entropy from state 1 to state 2 is positive, the stagnation pressure decreases from state 1 to state 2. Thus mass addition in a constant area duct results in a loss in stagnation pressure. The simple mass addition process path is shown in Fig. 7.

The example problem is typical of many mass addition situations. In common with the other simple flows, conditions at the sonic point represent both reference conditions and limits for the particular simple flow. For instance, simple mass addition is limited to a mass flow rate ratio, $\dot{m}/\dot{m}^*$, less than one. If the added mass results in a mass flow rate ratio greater than one, then the entrance conditions must change to accommodate the added mass flow.

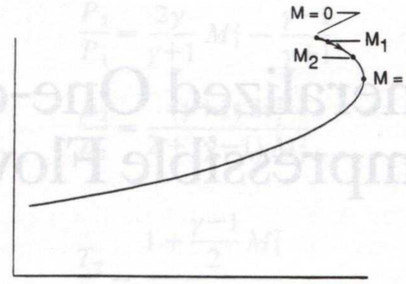

Fig. 7. Process path example.

CONCLUSIONS

The process path for simple mass addition is a meaningful representation for the process. Construction of the path on the Mollier diagram is very useful in delineating and understanding the characteristics and limiting behavior of simple mass addition. The process path has proven to be very helpful in introductory compressible flow courses and would seem to be a legitimate teaching/learning tool.

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